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# Evaluation of the Generalized $Q_M$ Function for Complex Arguments

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Several possibilities for computing the generalized $Q_M(a,b)$ function for complex arguments $a,b$ are presented and evaluated. Each approach has some limitations, such as loss of significance or complicated steepest paths of descent in an integral evaluation. However, a procedure is derived that will retain significance for all values of the parameters $a,b,$ while avoiding underflow or overflow, and the procedures has been automated. The need for this function arises when the joint probability density function of the $M-1$ largest squared envelopes of a matched filter output, plus the sum of the remainder, is considered. In particular, a Bromwich contour integral repeatedly utilizes this function in its evaluation.

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## TABLE OF CONTENTS

## LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS

| | |
|---|---|
| $a$ | Complex argument, equation (1) |
| $A,B$ | Auxiliary variables, equations (105) and (178) |
| angle | Principal angle function, equation (215) |
| $b$ | Complex argument, equation (1) |
| $c(u,\lambda)$ | Mixture function, equation (23) |
| $c_{\mathbf{s}}(u,\lambda)$ | Mixture function of **s**, equation (24) |
| $c_{\mathbf{e}}(u)$ | Cumulative distribution function of **e**, equation (20) |
| $C$ | Contour of integration, equation (64) |
| $C_s$ | Steepest descent contour, equation (137) |
| CDF | Cumulative distribution function |
| $D(z)$ | Denominator, equations (112) and (199) |
| $\mathbf{e}$ | Non-central chi variate, equation (18) |
| $e_{\mathbf{e}}(u)$ | Exceedance distribution function of **e**, equation (20) |
| $E$ | Auxiliary function, equation (75) |
| $E\{\ \}$ | Expectation, equation (3) |
| EDF | Exceedance distribution function |
| ES | Essential singularity |
| $f$ | Real positive constant used away from saddlepoint, equation (156) |
| $f_s$ | Real positive constant used at saddlepoint, equation (147) |
| $\mathbf{g}(k)$ | Gaussian random variable, equation (2) |
| $g(x)$ | Monotonic integrand, equation (85) |
| $g(z)$ | Auxiliary function, equation (211) |
| $h(x)$ | Integrand of similar shape, equation (86) |
| $h(\lambda,\sigma)$ | Auxiliary function, equations (27) and (56) |
| $I$ | Auxiliary integral, equation (75) |
| $J$ | Total number of random variables considered, equation (64) |
| $K$ | Number of Gaussian random variables, equation (2) |
| $K_n$ | Number of Gaussian random variables for case $n$, equation (51) |
| $m$ | Non-central mean parameter, equation (8) |
| $M$ | Integer associated with $Q_M(a,b)$ function, equations (1) and (10) |
| MGF | Moment-generating function |
| $N$ | Number of independent random variables, equation (51) |
| $N_a,N_b$ | Numbers of random variables with common statistics, equations (59) and (60) |
| $N(z)$ | Numerator, equations (111) and (198) |
| $p_{\mathbf{e}}(u)$ | Probability density function of **e**, equation (19) |
| $p_{\mathbf{s}}(u)$ | Probability density function of **s**, equation (9) |
| $P(z_1,\lambda)$ | Product of mixture functions, equation (65) |

## LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

| | |
|---|---|
| PDF | Probability density function |
| Prob | Probability, equation (14) |
| $q_2(z_1, z_2)$ | Joint probability density function, equation (64) |
| $q_M(a,b)$ | Complementary $Q_M(a,b)$ function, equation (20) |
| $Q_M(a,b)$ | Generalized $Q_M$ function, equation (1) |
| Re | Real part, equation (3) |
| Res | Residue, equation (181) |
| $\mathbf{s}$ | Non-central chi-squared random variable, equation (4) |
| $S_1, S_2, S_3$ | Auxiliary sums, equations (67) and (69) |
| SD | Steepest descent |
| SP | Saddlepoint |
| $U$ | Unit step function, equation (67) |
| $x_{H+}$ | Centers of hill locations, equation (215) |
| $x_{V+}$ | Centers of valley locations, equation (216) |
| $\mathbf{y}(k)$ | Squared Gaussian random variable, equation (2) |
| $z$ | Complex variable $z = x + i\ y$, equation (213) |
| $z_o$ | Location of essential singularity, equation (92) |
| $z_1, z_2$ | Field points for joint probability density function, equation (64) |
| $z_p$ | Location of pole, equation (208) |
| $z_s$ | Location of saddlepoint, equations (136), (185), and (207) |
| | |
| $\lambda$ | Argument of moment generating functions, equations (3) and (5) |
| $\mu(k)$ | Mean of $\mathbf{g}(k)$, equation (3) |
| $\mu_{\mathbf{s}}(\lambda)$ | Moment generating function of $\mathbf{s}$, equation (5) |
| $\mu_{\mathbf{y}(k)}(\lambda)$ | Moment generating function of $\mathbf{y}(k)$, equation (3) |
| $\theta$ | Perturbation angle, equations (151) and (162) |
| $\Phi(z)$ | Gaussian cumulative distribution function, equation (41) |
| $\sigma$ | Common standard deviation, equation (6) |
| $\sigma(k)$ | Standard deviation of $\mathbf{g}(k)$, equation (3) |
| $v$ | Auxiliary variable, equation (32) |
| $\phi(z)$ | Difference function, equation (138) |
| $\psi(z)$ | Logarithm of general integrand, equation (135) |
| $\Psi(z)$ | General integrand, equation (134) |

# EVALUATION OF THE GENERALIZED $Q_M$ FUNCTION FOR COMPLEX ARGUMENTS

## INTRODUCTION

The generalized $Q_M$ function is defined as (reference 1 and reference 2, section VII, equation 2.18)

$$Q_M(a,b) = \int_b^\infty dx\, x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2+a^2}{2}\right) I_{M-1}(a\,x). \tag{1}$$

Typically, $M$ is an integer, while parameters $a$ and $b$ are real and nonnegative. Numerous integrals involving the $Q_M(a,b)$ function are available in references 3 and 4. Also, a generalization to a general power of $(x/a)$, as well as a general order of the modified Bessel function $I_\nu$, was made in references 3 and 5. Numerical evaluation of $Q_M(a,b)$ for large values of arguments $a$ and/or $b$ is often plagued by underflow or overflow, due to the presence of exponentials with large positive or negative real arguments. Furthermore, in recent applications to active sonar processing (references 6 and 7), the need for evaluation of $Q_M(a,b)$ with complex arguments $a$ and $b$ was encountered; in particular, the joint probability density function of the $M-1$ largest envelope-squared values from a matched filter and the sum of the remainder required exactly this quantity $Q_M(a,b)$. However, the usual series expansions for $Q_M(a,b)$ can now contain powers of large complex numbers, thereby destroying any significance in the final result, due to cancellations of large positive and negative values of the real and imaginary components of the terms in the series.

To construct a viable routine for evaluation of $Q_M(a,b)$ with arbitrary complex arguments, a study of several possibilities was undertaken. The study begins with an investigation of the basic statistics of non-central chi-squared random variables (RVs), and progresses into the realm of probability density functions (PDFs), cumulative distribution functions (CDFs), and exceedance distribution functions (EDFs), where the $Q_M(a,b)$ function is encountered.

## STATISTICS OF NON-CENTRAL CHI-SQUARED RANDOM VARIABLES

Let $\{\mathbf{g}(k)\}$ be a set of independent Gaussian RVs with means $\{\mu(k)\}$ and standard deviations $\{\sigma(k)\}$, for $k = 1:K$. Then, the RVs

$$\mathbf{y}(k) = \mathbf{g}(k)^2 \quad \text{for } k = 1:K \tag{2}$$

have the moment-generating functions (MGFs)

$$\begin{aligned}\mu_{\mathbf{y}(k)}(\lambda) &= E\{\exp(\lambda\,\mathbf{y}(k))\} = E\{\exp(\lambda\,\mathbf{g}(k)^2)\} = \int du\, \exp(\lambda\, u^2)\, \frac{1}{\sqrt{2\pi}\,\sigma(k)} \exp\!\left(-\frac{(u-\mu(k))^2}{2\,\sigma(k)^2}\right) \\ &= (1-2\,\sigma(k)^2\,\lambda)^{-1/2} \exp\!\left(\frac{\mu(k)^2\,\lambda}{1-2\,\sigma(k)^2\,\lambda}\right) \quad \text{for } \mathrm{Re}(\lambda) < \frac{1}{2\,\sigma(k)^2},\ k = 1:K.\end{aligned} \tag{3}$$

The non-central chi-squared RV of interest here is given by the sum

$$\mathbf{s} = \sum_{k=1}^{K} \mathbf{y}(k) = \sum_{k=1}^{K} \mathbf{g}(k)^2. \tag{4}$$

The MGF of sum $\mathbf{s}$ follows from equation (3) and the independence of RVs $\{\mathbf{g}(k)\}$ as

$$\mu_{\mathbf{s}}(\lambda) = \prod_{k=1}^{K} (1-2\,\sigma(k)^2\,\lambda)^{-1/2} \quad \exp\!\left(\sum_{k=1}^{K} \frac{\mu(k)^2\,\lambda}{1-2\,\sigma(k)^2\,\lambda}\right) \quad \text{for } \mathrm{Re}(\lambda) < \frac{1}{2\max\{\sigma(k)^2\}}. \tag{5}$$

### SPECIAL CASE

Let the standard deviations of all the original Gaussian RVs $\{\mathbf{g}(k)\}$ be equal; that is,

$$\sigma(k) = \sigma \quad \text{for } k = 1:K. \tag{6}$$

Then, the MGF in equation (5) simplifies to

$$\mu_{\mathbf{s}}(\lambda) = (1-2\,\sigma^2\lambda)^{-K/2} \exp\!\left(\frac{m^2\,\lambda}{1-2\,\sigma^2\lambda}\right) \quad \text{for } \mathrm{Re}(\lambda) < \frac{1}{2\sigma^2}, \tag{7}$$

where

$$m^2 \equiv \sum_{k=1}^{K} \mu(k)^2. \tag{8}$$

Observe that the statistics of non-central chi-squared RV $\mathbf{s}$ depend only on the *sum* $m^2$ of the squares of the means, $\{\mu(k)\}$, of the individual Gaussian RVs $\{\mathbf{g}(k)\}$ when equation (6) is true.

The PDF of RV $\mathbf{s}$, corresponding to MGF (7), is

$$p_{\mathrm{s}}(u) = p(u;K,m,\sigma) = \frac{1}{2\sigma^2}\left(\frac{\sqrt{u}}{m}\right)^{M-1} \exp\left(-\frac{u+m^2}{2\sigma^2}\right) I_{M-1}\left(\frac{m}{\sigma^2}\sqrt{u}\right) \quad \text{for } u>0,\ m>0, \tag{9}$$

where

$$M \equiv \frac{K}{2}, \quad m = \left(\sum_{k=1}^{K} \mu(k)^2\right)^{1/2} . \tag{10}$$

$M$ need not be an integer. A better numerical alternative to equation (9) is

$$p_{\mathrm{s}}(u) = \frac{1}{2\sigma^2}\left(\frac{\sqrt{u}}{m}\right)^{M-1} \exp\left(-\frac{(\sqrt{u}-m)^2}{2\sigma^2}\right) \exp\left(-\frac{m\sqrt{u}}{\sigma^2}\right) I_{M-1}\left(\frac{m\sqrt{u}}{\sigma^2}\right) \quad \text{for } u>0,\ m>0, \tag{11}$$

because the well-behaved combination

$$\exp(-x)\, I_{M-1}(x) \equiv \text{besseli}(M-1,x,1) \tag{12}$$

is directly available as a MATLAB function. For $m = 0$, equation (11) reduces to

$$p_{\mathrm{s}}(u) = \frac{u^{M-1}}{\Gamma(M)\,(2\sigma^2)^M} \exp\left(-\frac{u}{2\sigma^2}\right) \quad \text{for } u>0. \tag{13}$$

The EDF for RV $\mathbf{s}$ in equation (4) is, upon use of equations (9) and (1),

$$e_{\mathrm{s}}(u) = \text{Prob}(\mathbf{s}>u) = \int_u^\infty dt\ p_{\mathrm{s}}(t) = Q_M\left(\frac{m}{\sigma},\frac{\sqrt{u}}{\sigma}\right) \quad \text{for } u>0. \tag{14}$$

The $j$-th cumulant of RV $\mathbf{s}$ is

$$\chi_{\mathrm{s}}(j) = j!\left(\frac{M}{j}+\frac{m^2}{2\sigma^2}\right)(2\sigma^2)^j \quad \text{for } j = 1,2,\ldots, \tag{15}$$

while the $\nu$-th moment of RV $\mathbf{s}$ is

$$E\{\mathbf{s}^{\nu}\} = (2\sigma^2)^{\nu}\,\frac{\Gamma(M+\nu)}{\Gamma(M)}\;{}_1F_1\!\left(-\nu; M; -\frac{m^2}{2\sigma^2}\right) \tag{16}$$

in terms of a confluent hypergeometric function. For $\nu$ equal to an integer $j$, this result simplifies to

$$E\{\mathbf{s}^{j}\} = j!\,(2\sigma^2)^{j}\,L_j^{(M-1)}\!\left(-\frac{m^2}{2\sigma^2}\right), \tag{17}$$

where the latter function is a Laguerre polynomial.

An alternative RV of interest is the square root of the sum of squares of the Gaussian RVs, namely, non-central chi-variate

$$\mathbf{e} = \sqrt{\mathbf{s}} = \sqrt{\sum_{k=1}^{K} \mathbf{g}(k)^2}\;. \tag{18}$$

For the case of equal Gaussian standard deviations, equation (6), the PDF of RV $\mathbf{e}$ follows from equation (9) as

$$p_{\mathbf{e}}(u) = 2u\,p_{\mathbf{s}}(u^2) = \frac{u}{\sigma^2}\left(\frac{u}{m}\right)^{M-1}\exp\!\left(-\frac{u^2+m^2}{2\sigma^2}\right) I_{M-1}\!\left(\frac{mu}{\sigma^2}\right) \quad \text{for } u>0,\; m>0. \tag{19}$$

The corresponding EDF and CDF are, respectively,

$$e_{\mathbf{e}}(u) = Q_M\!\left(\frac{m}{\sigma},\frac{u}{\sigma}\right), \quad c_{\mathbf{e}}(u) = 1 - e_{\mathbf{e}}(u) = 1 - Q_M\!\left(\frac{m}{\sigma},\frac{u}{\sigma}\right) \equiv q_M\!\left(\frac{m}{\sigma},\frac{u}{\sigma}\right) \quad \text{for } u>0, \tag{20}$$

where $q_M$ is the complementary $Q_M$ function. A better numerical alternative to equation (19) is

$$p_{\mathbf{e}}(u) = \frac{u}{\sigma^2}\left(\frac{u}{m}\right)^{M-1}\exp\!\left(-\frac{(u-m)^2}{2\sigma^2}\right)\exp\!\left(-\frac{mu}{\sigma^2}\right) I_{M-1}\!\left(\frac{mu}{\sigma^2}\right) \quad \text{for } u>0,\; m>0; \tag{21}$$

see equation (12). For $m=0$, equation (21) reduces to

$$p_{\mathbf{e}}(u) = \frac{2u^{2M-1}}{\Gamma(M)\,(2\sigma^2)^M}\exp\!\left(-\frac{u^2}{2\sigma^2}\right) \quad \text{for } u>0. \tag{22}$$

## DETERMINATION OF THE MIXTURE FUNCTION

A mixture of a CDF and an MGF was defined in reference 6, equation (3), namely,

$$c(u,\lambda) = \int_{-\infty}^{u} dx \; p(x) \exp(\lambda x), \tag{23}$$

where $p(x)$ is a PDF. That is, $c(u,0)$ is the usual CDF, while $c(+\infty,\lambda)$ is the usual MGF. Variable $u$ is real, while $\lambda$ can be complex. For the non-central chi-squared RV $\mathbf{s}$ defined in equation (4), the mixture function is given by equations (23) and (9) as

$$\begin{aligned} c_s(u,\lambda) &= \int_0^u dx \; p_s(x) \exp(\lambda x) \\ &= \frac{\exp(-r^2/2)}{2\sigma^2 m^{M-1}} \int_0^u dx \; x^{(M-1)/2} \exp\left(-x\left(\frac{1}{2\sigma^2} - \lambda\right)\right) I_{M-1}\left(\frac{m\sqrt{x}}{\sigma^2}\right) \quad \text{for } u > 0, \end{aligned} \tag{24}$$

where

$$M = \frac{K}{2}, \quad r \equiv \frac{m}{\sigma} = \frac{1}{\sigma}\sqrt{\sum_{k=1}^{K} \mu(k)^2} \; . \tag{25}$$

Upon making the substitution $x = t^2$ in equation (24), there follows

$$\begin{aligned} c_s(u,\lambda) = c(u,\lambda;K,m,\sigma) &= \frac{\exp(-r^2/2)}{\sigma^2 m^{M-1}} \int_0^{\sqrt{u}} dt \; t^M \exp\left(-t^2\left(\frac{1}{2\sigma^2} - \lambda\right)\right) I_{M-1}\left(\frac{mt}{\sigma^2}\right) \\ &= \frac{1}{h^{2M}} \exp\left(-\frac{r^2}{2}\left(1 - \frac{1}{h^2}\right)\right) q_M\left(\frac{r}{h}, \frac{h}{\sigma}\sqrt{u}\right) \quad \text{for } all\ \lambda \neq \frac{1}{2\sigma^2}, \; u > 0, \end{aligned} \tag{26}$$

which follows directly from definitions (1) and (20), and auxiliary definitions

$$M = \frac{K}{2}, \quad r = \frac{m}{\sigma} = \frac{1}{\sigma}\sqrt{\sum_{k=1}^{K} \mu(k)^2} \; , \quad h = h(\lambda,\sigma) = \sqrt{1 - 2\sigma^2 \lambda}. \tag{27}$$

The special case of $\lambda = \dfrac{1}{2\sigma^2}$ is presented later; see equation (57).

As partial checks on relation (26), there follows $c_s(0,\lambda) = 0$, upon use of $q_M(a,0) = 0$. Also, $c_s(u,0) = q_M(r,\sqrt{u}/\sigma) = 1 - e_s(u)$, which is in agreement with equation (20). Finally,

$c_s(+\infty,\lambda)$ reduces to the MGF in equation (7), using $q_M(a,+\infty)=1$. Result (26) has been checked numerically against the integral relation (24).

Another quantity of interest is

$$c_s(u,\lambda,1) \equiv \frac{\partial}{\partial\lambda} c_s(u,\lambda) = \int_0^u dx\, x\, p_s(x)\exp(\lambda x), \tag{28}$$

by reference to equation (24). By making the change of variable $x=t^2$ and referring to the top line of equation (26), there follows

$$c_s(u,\lambda,1) = \frac{\exp(-r^2/2)}{\sigma^2 m^{M-1}} \int_0^{\sqrt{u}} dt\, t^{M+2} \exp\left(-t^2\left(\frac{1}{2\sigma^2}-\lambda\right)\right) I_{M-1}\left(\frac{mt}{\sigma^2}\right). \tag{29}$$

At this point, the integral result

$$\begin{aligned}\int_0^b dx\, x^{M+2} \exp\left(-\frac{p^2x^2}{2}\right) I_{M-1}(ax) = {}& \frac{a^{M-1}}{p^{2M+4}}(a^2+2Mp^2)\exp\left(\frac{a^2}{2p^2}\right) q_M\left(\frac{a}{p},bp\right) \\ & -\frac{b^M}{p^4}\exp\left(-\frac{p^2b^2}{2}\right)\left[a\, I_M(ab)+b\,p^2\, I_{M-1}(ab)\right]\end{aligned} \tag{30}$$

is used, with the end result (after some manipulations)

$$\begin{aligned}c_s(u,\lambda,1) = \frac{\sigma^2}{h^4}\exp\left(-\frac{r^2}{2}\right)\Bigg\{ & \frac{r^2+2Mh^2}{h^{2M}}\exp\left(\frac{r^2}{2h^2}\right) q_M\left(\frac{r}{h},hv\right) \\ & -\frac{v^M}{r^{M-1}}\exp\left(-\frac{h^2v^2}{2}\right)\left[r\, I_M(rv)+h^2\, v\, I_{M-1}(rv)\right]\Bigg\} \quad \text{for } \lambda \neq \frac{1}{2\sigma^2},\ u>0,\ m>0,\end{aligned} \tag{31}$$

with

$$M=\frac{K}{2},\ r=\frac{m}{\sigma}=\frac{1}{\sigma}\sqrt{\sum_{k=1}^K \mu(k)^2},\ h=h(\lambda,\sigma)=\sqrt{1-2\sigma^2\lambda},\ v=\frac{\sqrt{u}}{\sigma}. \tag{32}$$

The result in equation (31) has been checked numerically against the integral relation (29).

## SADDLEPOINT LOCATION FOR MGF (7)

The logarithm of MGF $\mu_s(\lambda)$ in equation (7) is

$$\chi_s(\lambda) = \log \mu_s(\lambda) = -\frac{K}{2}\log(1-2\sigma^2\lambda) + \frac{m^2\lambda}{1-2\sigma^2\lambda} \quad \text{for } \mathrm{Re}(\lambda) < \frac{1}{2\sigma^2}. \tag{33}$$

The derivative of this quantity is

$$\chi_s'(\lambda) = K\sigma^2 y + m^2 y^2, \quad y \equiv \frac{1}{1-2\sigma^2\lambda}. \tag{34}$$

The saddlepoint (SP) location $\lambda_o$ is given by the solution of the equation

$$x = \chi_s'(\lambda_o) = K\sigma^2 y_o + m^2 y_o^2, \tag{35}$$

where $x$ is the field point of interest in the PDF $p_s$. Solving the quadratic equation (35) for $y_o$ and then solving the second term of equation (34) for the SP gives

$$\lambda_o = \lambda_o(x) = \frac{1}{2\sigma^2}\left(1 - \frac{2m^2}{\sqrt{K^2\sigma^4 + 4m^2 x} - K\sigma^2}\right) = \frac{1}{2\sigma^2}\left(1 - \frac{\sqrt{K^2\sigma^4 + 4m^2 x} + K\sigma^2}{2x}\right). \tag{36}$$

This is an explicit relation for the SP location in terms of field point $x$. As $x$ tends to zero, $\lambda_o$ tends to $-\infty$, while as $x$ gets large, $\lambda_o$ tends to $1/(2\sigma^2)-$. The actual saddlepoint approximation to the non-central chi-squared PDF at field point $x$ is

$$p_s(x) \cong \frac{\mu_s(\lambda_o)\exp(-\lambda_o x)}{\sqrt{2\pi\,\chi_s''(\lambda_o)}}, \quad \lambda_o = \lambda_o(x), \tag{37}$$

where

$$\chi_s''(\lambda) = 2K\sigma^4 y^2 + 4m^2\sigma^2 y^3 \tag{38}$$

from equation (34).

## EVALUATION OF FUNCTION $Q_M(a,b)$ FOR $M$ A SMALL HALF-INTEGER

The evaluation of $Q_M(a,b)$, defined in equation (1), is simpler when $M$ is a half-integer, that is, when the number $K$ of RVs added, is odd; see equations (4) and (10). For $M = 1/2$, the Bessel function in equation (1) is given by reference 8, equation 10.2.14, as

$$I_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cosh(z). \tag{39}$$

Substitution in equation (1) immediately leads to the result

$$Q_{1/2}(a,b) = \Phi(a-b) + \Phi(-a-b), \quad q_{1/2}(a,b) = 1 - Q_{1/2}(a,b) = \Phi(b-a) - \Phi(-a-b), \tag{40}$$

where

$$\Phi(z) \equiv (2\pi)^{-1/2} \int_{-\infty}^{z} dt\, \exp(-t^2/2) = (2\pi)^{-1/2} \int_{-z}^{\infty} dt\, \exp(-t^2/2) \tag{41}$$

is the normalized Gaussian CDF. The values of $Q_M(a,b)$ for $M = 1.5, 2.5, 3.5 \ldots$ can then be found from reference 4, equation (3), top line, which does not require $M$ to be an integer. The required values of the Bessel functions are available in reference 8, equation 10.2.13. The end results are

$$\begin{aligned}
Q_{3/2}(a,b) &= Q_{1/2}(a,b) + \sqrt{\frac{2}{\pi}} \exp\left(-\frac{a^2+b^2}{2}\right) \frac{\sinh(ab)}{a}, \\
Q_{5/2}(a,b) &= Q_{3/2}(a,b) + \sqrt{\frac{2}{\pi}} \exp\left(-\frac{a^2+b^2}{2}\right) \frac{ab\cosh(ab) - \sinh(ab)}{a^3}, \\
Q_{7/2}(a,b) &= Q_{5/2}(a,b) + \sqrt{\frac{2}{\pi}} \exp\left(-\frac{a^2+b^2}{2}\right) \frac{(3+a^2b^2)\sinh(ab) - 3ab\cosh(ab)}{a^5}.
\end{aligned} \tag{42}$$

The quantity $q_M(a,b)$ appears in equations (26) and (31). Then, numerically, the second result in equation (40) is more appropriate, and the top line of equation (42) can be modified to

$$q_{3/2}(a,b) = q_{1/2}(a,b) - \sqrt{\frac{2}{\pi}} \exp\left(-\frac{a^2+b^2}{2}\right) \frac{\sinh(ab)}{a}, \tag{43}$$

along with obvious changes to the other two relations. This procedure avoids the subtraction of $Q_M$ from unity and retains more accurate numerical results.

All three results in equation (42) have removable singularities at $a = 0$. The special values there are

$$
\begin{aligned}
Q_{3/2}(0,b) &= 2\,\Phi(-b) + \sqrt{\frac{2}{\pi}}\exp\left(-\frac{b^2}{2}\right) b, \\
Q_{5/2}(0,b) &= Q_{3/2}(0,b) + \sqrt{\frac{2}{\pi}}\exp\left(-\frac{b^2}{2}\right)\frac{b^3}{3}, \\
Q_{7/2}(0,b) &= Q_{5/2}(0,b) + \sqrt{\frac{2}{\pi}}\exp\left(-\frac{b^2}{2}\right)\frac{b^5}{15}.
\end{aligned}
\tag{44}
$$

However, the last two results in equation (42) have numerical problems when product $ab$ is near zero, due to the differences of nearly-equal quantities in the numerators. Power series expansions are required for the last two results in equation (42); they are

$$
\begin{aligned}
z\cosh(z) - \sinh(z) &\cong \frac{z^3}{3} + \frac{z^5}{30} + \frac{z^7}{840} + \frac{z^9}{45360} + \frac{z^{11}}{3991680} + \frac{z^{13}}{518918400}, \\
(3+z^2)\sinh(z) - 3z\cosh(z) &\cong \frac{z^5}{15} + \frac{z^7}{210} + \frac{z^9}{7560} + \frac{z^{11}}{498960} + \frac{z^{13}}{51891840} + \frac{z^{15}}{7783776000}.
\end{aligned}
\tag{45}
$$

A MATLAB program, QM_half_integer(M,a,b), has been written that incorporates all these features, at least up to $M = 7/2$. Although a recursion based on reference 4, equation (3), may seem attractive, it is not useful for small $a$, as may be seen from the low-order results in equation (42) above, where successively higher powers of parameter $a$ appear in the denominator. The recursion inevitably encounters differences of like quantities, which are then divided by very small quantities, resulting in severe losses of significance in the final results. Nothing in a recursion can avoid the required switch to a power series (45) for small arguments.

## SERIES EVALUATION OF $q_M(a,b)$ FOR $M$ INTEGER

The series expansion

$$
Q_M(a,b) = \exp(-A-B)\sum_{k=0}^{\infty}\frac{A^k}{k!}\sum_{n=0}^{k+M-1}\frac{B^n}{n!}, \quad A = \frac{a^2}{2}, B = \frac{b^2}{2},
\tag{46}
$$

was derived in reference 4, equation (8), by expanding the Bessel function $I_{M-1}(ax)$ in equation (1) in a power series and integrating term by term. By expressing the inner sum on $n$ as the difference between a sum to infinity and a sum from $k + M$ to infinity, equation (46) easily yields

$$q_M(a,b) = 1 - Q_M(a,b) = \exp(-A-B) \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{n=k+M}^{\infty} \frac{B^n}{n!}. \tag{47}$$

Upon interchanging the two summations, the following alternatives are obtained:

$$q_M(a,b) = \exp(-A-B) \sum_{n=M}^{\infty} \frac{B^n}{n!} \sum_{k=0}^{n-M} \frac{A^k}{k!} = \exp(-A-B) \frac{B^M}{M!} \sum_{j=0}^{\infty} \frac{B^j}{(M+1)_j} \sum_{k=0}^{j} \frac{A^k}{k!}. \tag{48}$$

The last form involves only one infinite sum, all the terms of which are very amenable to evaluation by way of recursions. Also, for $a$ and $b$ real, every quantity is nonnegative. However, if $a$ or $b$ is complex, positive and negative quantities will occur in equation (48), and all significance can be lost in the final sum. Equation (27) indicates that if $\lambda$ is complex, then $h = h(\lambda, \sigma)$ will be complex, and the $q_M(a,b)$ term in equation (26) will have to be evaluated for complex arguments $a$ and $b$.

When $|b|$ is large relative to $|a|$, a useful alternative to the above series expansions is afforded by equation (4) of reference 4:

$$Q_M(a,b) = \left(\frac{b}{a}\right)^{M-1} \exp\left(-\frac{1}{2}(b-a)^2\right) \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n \exp(-ab)\, I_{n+1-M}(ab). \tag{49}$$

The magnitudes of both terms in the sum eventually decrease with $n$; see reference 8, page 428. The useful combination in equation (12) is encountered again in this form.

Conversely, when $|a|$ is larger than $|b|$, equation (5) of reference 4 yields

$$q_M(a,b) = \left(\frac{b}{a}\right)^{M} \exp\left(-\frac{1}{2}(b-a)^2\right) \sum_{n=0}^{\infty} \left(\frac{b}{a}\right)^n \exp(-ab)\, I_{n+M}(ab). \tag{50}$$

## JOINT PROBABILITY DENSITY FUNCTION OF $J-1$ LARGEST RANDOM VARIABLES AND THE SUM OF THE REMAINDER OF A SET OF $N$ INDEPENDENT RANDOM VARIABLES

The joint PDF of the $J-1$ largest RVs and the sum of the remainder of a set of $N$ independent RVs $\{\mathbf{s}_n\}$ is presented in reference 7—in particular, see equations (15), (28), (38), and (47), for $J = 2, 3, 4$, and 5, respectively. The PDFs $\{p_n(x)\}$ of the $N$ independent RVs $\{\mathbf{s}_n\}$ are arbitrary in all these cases.

### NON-CENTRAL CHI-SQUARED RANDOM VARIABLES

Let the $n$-th independent RV $\mathbf{s}_n$ be given by

$$\mathbf{s}_n = \sum_{k=1}^{K_n} \mathbf{g}_n(k)^2 \quad \text{for } n = 1:N, \tag{51}$$

where independent Gaussian RVs

$$\mathbf{g}_n(k) = \text{Normal}\{\mu_n(k), \sigma_n\} \quad \text{for } k = 1:K_n,\ \ n = 1:N. \tag{52}$$

Notice that standard deviation $\sigma_n$ is independent of $k$. Then, as shown in equation (9), the PDF of RV $\mathbf{s}_n$ is

$$p_n(u) \equiv p_{\mathbf{s}_n}(u) = \frac{1}{2\sigma_n^2}\left(\frac{t_n}{r_n}\right)^{M_n-1} \exp\left(-\frac{t_n^2 + r_n^2}{2}\right) I_{M_n-1}(r_n\, t_n) \quad \text{for } u > 0,\ \ m_n > 0, \tag{53}$$

where

$$M_n = \frac{K_n}{2},\ \ m_n = \left(\sum_{k=1}^{K_n} \mu_n(k)^2\right)^{1/2},\ \ r_n = \frac{m_n}{\sigma_n},\ \ t_n = \frac{\sqrt{u}}{\sigma_n} \quad \text{for } n = 1:N. \tag{54}$$

Also, from equation (26), the mixture function of RV $\mathbf{s}_n$ is

$$c_n(u,\lambda) = \frac{1}{h_n(\lambda)^{2M_n}} \exp\left(-\frac{r_n^2}{2}\left(1 - \frac{1}{h_n(\lambda)^2}\right)\right) q_{M_n}\left(\frac{r_n}{h_n(\lambda)}, \frac{h_n(\lambda)}{\sigma_n}\sqrt{u}\right) \text{ for } all\ \lambda \neq \frac{1}{2\sigma_n^2}, u > 0, \tag{55}$$

where

$$h_n(\lambda) = h(\lambda, \sigma_n) = \sqrt{1 - 2\sigma_n^2\, \lambda} \quad \text{for } n = 1:N. \tag{56}$$

For $\lambda = \dfrac{1}{2\sigma_n^2}$, $h_n(\lambda) = 0$ and equation (55) reduces to

$$c_n\left(u, \frac{1}{2\sigma_n^2}\right) = \exp\left(-\frac{r_n^2}{2}\right)\left(\frac{1}{r_n}\frac{\sqrt{u}}{\sigma_n}\right)^{M_n} I_{M_n}\left(r_n \frac{\sqrt{u}}{\sigma_n}\right), \; u > 0, \; m_n > 0. \tag{57}$$

For $m_n = 0$, this reduces further to

$$c_n\left(u, \frac{1}{2\sigma_n^2}\right) = \frac{1}{M!}\left(\frac{u}{2\sigma_n^2}\right)^M, \; u > 0. \tag{58}$$

Interest is centered here on the situation where $N_a$ of the $K_a$ sets of $N$ independent Gaussian RVs in equation (52) have a common mean and standard deviation, namely,

$$\mathbf{g}_n(k) = \text{Normal}\{\mu_a(k), \sigma_a\} \;\text{ for }\; k = 1:K_a, \; n = 1:N_a, \tag{59}$$

while the remaining $K_b$ sets of $N_b = N - N_a$ independent Gaussian RVs have a different common mean and standard deviation:

$$\mathbf{g}_n(k) = \text{Normal}\{\mu_b(k), \sigma_b\} \;\text{ for }\; k = 1:K_b, \; n = N_a + 1:N. \tag{60}$$

Then, from equation (8),

$$m_a = \left(\sum_{k=1}^{K_a} \mu_a(k)^2\right)^{1/2}, \; m_b = \left(\sum_{k=1}^{K_b} \mu_b(k)^2\right)^{1/2}. \tag{61}$$

The corresponding PDF and mixture function of RVs $\{\mathbf{s}_n\}$, $n = 1:N_a$, are, respectively,

$$\begin{aligned} p_a(u) = p(u; K_a, m_a, \sigma_a) &\equiv \frac{1}{2\sigma_a^2}\left(\frac{\sqrt{u}}{\sigma_a r_a}\right)^{M_a - 1} \exp\left(-\frac{u}{2\sigma_a^2} - \frac{r_a^2}{2}\right) I_{M_a - 1}\left(\frac{r_a}{\sigma_a}\sqrt{u}\right), \\ c_a(u, \lambda) = c(u, \lambda; K_a, m_a, \sigma_a) &\equiv \frac{1}{h_a(\lambda)^{2M_a}} \exp\left(-\frac{r_a^2}{2}\left(1 - \frac{1}{h_a(\lambda)^2}\right)\right) q_{M_a}\left(\frac{r_a}{h_a(\lambda)}, \frac{h_a(\lambda)}{\sigma_a}\sqrt{u}\right), \end{aligned} \tag{62}$$

for $u > 0$ and $n = 1:N_a$, where

$$M_a = \frac{K_a}{2}, \; r_a = \frac{m_a}{\sigma_a}, \; h_a(\lambda) = h(\lambda, \sigma_a) = \sqrt{1 - 2\sigma_a^2 \lambda}. \tag{63}$$

For RVs $\{\mathbf{s}_n\}$, $n = N_a + 1:N$, replace all the subscripts $a$ by $b$ in equations (62) and (63).

## JOINT PDF OF LARGEST RANDOM VARIABLE AND SUM OF REMAINDER

The first case of interest is given by equations (15) and (16) of reference 7; namely, the joint PDF is, for $J = 2$,

$$q_2(z_1,z_2) = \frac{1}{i2\pi} \int_C d\lambda \, \exp(-\lambda z_2) \, P(z_1,\lambda) \sum_{n=1}^{N} \frac{p_n(z_1)}{c_n(z_1,\lambda)}, \tag{64}$$

where product

$$P(z_1,\lambda) = \prod_{n=1}^{N} c_n(z_1,\lambda) = c_a(z_1,\lambda)^{N_a} \, c_b(z_1,\lambda)^{N_b} \tag{65}$$

and sum

$$\sum_{n=1}^{N} \frac{p_n(z_1)}{c_n(z_1,\lambda)} = N_a \frac{p_a(z_1)}{c_a(z_1,\lambda)} + N_b \frac{p_b(z_1)}{c_b(z_1,\lambda)}. \tag{66}$$

Bromwich contour $C$ in equation (64) runs from $-i\infty$ to $+i\infty$ in the complex $\lambda$ plane. For numerical accuracy, this contour $C$ can be moved so as to pass through the (unique real) SP $\lambda_s$ of the integrand of equation (64); of course, this SP location depends on the particular two-dimensional field point $(z_1,z_2)$ of interest, that is, $\lambda_s = \lambda_s(z_1,z_2)$. This SP location must be recalculated every time the two-dimensional field point is changed.

## JOINT PDF OF TWO LARGEST RANDOM VARIABLES AND SUM OF REMAINDER

This result is available from reference 7, equations (28) through (30); namely, the joint PDF is, for $J = 3$,

$$q_3(z_1,z_2,z_3) = U(z_1 - z_2) \frac{1}{i2\pi} \int_C d\lambda \, \exp(-\lambda z_3) \, P(z_2,\lambda) \, (S_1 \, S_2 - S_3), \tag{67}$$

where product

$$P(z_2,\lambda) = \prod_{n=1}^{N} c_n(z_2,\lambda) = c_a(z_2,\lambda)^{N_a} \, c_b(z_2,\lambda)^{N_b} \tag{68}$$

and sums

$$
\begin{aligned}
S_1 &= N_a \frac{p_a(z_1)}{c_a(z_2,\lambda)} + N_b \frac{p_b(z_1)}{c_b(z_2,\lambda)}, \quad S_2 = N_a \frac{p_a(z_2)}{c_a(z_2,\lambda)} + N_b \frac{p_b(z_2)}{c_b(z_2,\lambda)}, \\
S_3 &= N_a \frac{p_a(z_1)\, p_a(z_2)}{c_a(z_2,\lambda)^2} + N_b \frac{p_b(z_1)\, p_b(z_2)}{c_b(z_2,\lambda)^2}.
\end{aligned}
\tag{69}
$$

The comments under equation (66) are again directly relevant, except that the field point $(z_1, z_2, z_3)$ in equation (67) is now three-dimensional.

The extensions to larger values of $J$ are given in reference 7, equations (38) through (42) for $J = 4$, equations (47) through (50) for $J = 5$, and equation (51) for the general case.

## EVALUATION OF THE $Q$ FUNCTION FOR ALL REAL ARGUMENTS

In this section, attention is limited to the case $M = 1$. Then, equation (1) reduces to

$$Q(a,b) = \int_b^\infty dx\, x \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(a\,x). \tag{70}$$

Although this function is analytic in each of the variables $a$ and $b$, initial interest here is in numerical evaluation of $Q(a,b)$ only for nonnegative real $a,b$ for all ranges of their values. One of the problems associated with this evaluation is that underflow and/or overflow frequently occur for large arguments, and all significance is lost. A method will be presented that isolates the underflow/overflow problem and allows for the calculation of $Q$ or $\log(Q)$ for all arguments. Even when $Q$ underflows or overflows the capability of the computer, $\log(Q)$ still retains significance. This situation is analogous to the gamma function $\Gamma(x)$, where both $\Gamma(x)$ and $\log(\Gamma(x))$ are furnished by a computer, to circumvent overflow.

In addition, the complementary $Q$ function now becomes

$$q(a,b) = 1 - Q(a,b) = \int_0^b dx\, x \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(a\,x). \tag{71}$$

When $Q$ approaches 1 so closely that all significance is lost, the complementary $q$ function still retains significance. Again, since $q$ could also underflow, there is interest in computing both $q$ and $\log(q)$ for all $a,b$. This situation is similar to that where both the error function, erf(x), and the complementary error function, erfc(x) = 1–erf(x), are furnished by a computer. The $Q$ function is an exceedance probability, while the $q$ function is a cumulative probability of a normalized Rician random variate, as noted in an earlier section.

In reference 3, equation (74), the following integral result was presented:

$$\int_0^\pi dx \frac{1}{\pi} \exp\left(-\frac{\alpha}{1-\beta\cos(x)}\right) = 2\,Q\left(\frac{1}{u}\sqrt{\alpha\,(1-u)}, \frac{1}{u}\sqrt{\alpha\,(1+u)}\right) - \exp\left(-\frac{\alpha}{1-\beta^2}\right) I_0\left(\frac{\alpha\,\beta}{1-\beta^2}\right) \tag{72}$$

for $\beta < 1$, with $u = \sqrt{1-\beta^2}$. By making appropriate changes of variables, this result can be manipulated into the following forms (reference 5, equation 18)

$$Q(a,b) = I + E, \quad q(a,b) = 1 - Q(a,b) \ \text{ for } a \le b, \tag{73}$$

and

$$q(a,b) = I - E, \quad Q(a,b) = 1 - q(a,b) \quad \text{for } a \geq b, \tag{74}$$

where

$$I = \frac{1}{2\pi}\int_0^\pi dx \exp\left(-\frac{c_1}{1-c_2\cos(x)}\right), \quad c_1 = \frac{(a^2-b^2)^2}{2(a^2+b^2)}, \quad c_2 = \frac{2ab}{a^2+b^2},$$
$$E = \frac{1}{2}\exp\left(-\frac{a^2+b^2}{2}\right) I_0(a\,b). \tag{75}$$

A useful alternative to the latter term is

$$E = \exp\left(-\frac{(a-b)^2}{2}\right)\frac{1}{2}\exp(-a\,b)\, I_0(a\,b), \tag{76}$$

in that the product $\exp(-x)\,I_0(x)$ is very well behaved for all real nonnegative $x$, decaying as $1/\sqrt{2\pi x}$ for large $x$. In fact, this product is denoted as besseli(0,x,1) in MATLAB. Thus, all the underflow problems associated with $E$ are isolated in the leading exponential.

At the same time, integral $I$ in equation (75) can be expressed as

$$I = \exp\left(-\frac{c_1}{1+c_2}\right)\frac{1}{2\pi}\int_0^\pi dx \exp\left(-\frac{c_1\,c_2}{1+c_2}\,\frac{1+\cos(x)}{1-c_2\cos(x)}\right), \tag{77}$$

for which the maximum value of the integrand is 1 at $x = \pi$. Now, by employing the definitions in equation (75), this expression takes the form

$$I = \exp\left(-\frac{(a-b)^2}{2}\right)\frac{1}{2\pi}\int_0^\pi dx \exp\left(-\frac{a\,b\,(a-b)^2}{a^2+b^2}\,\frac{1+\cos(x)}{1-c_2\cos(x)}\right). \tag{78}$$

Again, the *exact same* leading exponential accounts for all the underflow problems associated with $I$. The integrand of equation (78) increases monotonically over the interval $(0,\pi)$, ending up at value 1, and cannot lead to underflow.

For computational purposes, define the parameters

$$t_1 = \frac{1}{2}(a-b)^2, \quad t_2 = a\,b, \quad t_3 = \frac{2\,t_2}{a^2+b^2}, \quad t_4 = t_1\,t_3. \tag{79}$$

Then, from equations (78) and (76), there follows

$$I = \exp(-t_1)\,\frac{1}{2\pi}\int_0^{\pi} dx\,\exp\left(-t_4\,\frac{1+\cos(x)}{1-t_3\,\cos(x)}\right),$$
$$E = \exp(-t_1)\,\frac{1}{2}\exp(-t_2)\,I_0(t_2). \tag{80}$$

Therefore, the two primary terms needed in equations (73) and (74) can be written as

$$I \pm E = \exp(-t_1)\left[\frac{1}{2\pi}\int_0^{\pi} dx\,\exp\left(-t_4\,\frac{1+\cos(x)}{1-t_3\,\cos(x)}\right) \pm \frac{1}{2}\exp(-t_2)\,I_0(t_2)\right], \tag{81}$$

where all the underflow issues are concentrated in one term, $\exp(-t_1)$. This immediately leads to

$$\log(I \pm E) = -t_1 + \log\left[\frac{1}{2\pi}\int_0^{\pi} dx\,\exp\left(-t_4\,\frac{1+\cos(x)}{1-t_3\,\cos(x)}\right) \pm \frac{1}{2}\exp(-t_2)\,I_0(t_2)\right], \tag{82}$$

which will not underflow for *any* $a,b$. The complementary quantities in equations (73) and (74) are then available upon subtraction of equation (81) from 1. At least one of the four quantities

$$Q,\ q,\ \log(Q),\ \log(q) \tag{83}$$

will retain significance for all values of $a,b$.

A MATLAB routine was written for evaluation of the four quantities in equation (83), and is called according to

$$[Q, q, Q\log, q\log] = Qq\log(a,b) \tag{84}$$

The time-consuming portion of this procedure is the numerical evaluation of the integral in equation (82). The major problem is that the monotonic integrand in equation (82), namely,

$$g(x) \equiv \exp\left(-t_4\,\frac{1+\cos(x)}{1-t_3\,\cos(x)}\right), \tag{85}$$

varies from $g(0) = \exp(-2\,t_4/(1-t_3)) = \exp(-2\,a\,b)$ to $g(\pi) = 1$, and can do so in a very narrow interval in $x$. This can sometimes cause the routine in equation (84) to expend a great deal of time achieving a stable estimate of the integral required in equation (82). Integrand $g(x)$ does not have a peak anywhere in $(0,\pi)$; however, $g(x)$ can have a narrow transition range for its values, which impedes quick evaluation of the integral. This sampling problem has been circumvented by resorting to MATLAB's quad or quad$\ell$ routines, which employ an *adaptive* Simpson's rule that resorts to finer sampling where the integrand varies most rapidly.

In an effort to ameliorate this rapid transition of $g(x)$, an integrand of similar shape was constructed, namely,

$$h(x) \equiv \exp\left(-t_4 \frac{1+\cos(x)}{1-\cos(x)}\right) = \exp(t_4) \exp\left(-\frac{2\,t_4}{1-\cos(x)}\right). \tag{86}$$

Its integral is available in closed form (reference 3, equation (75)):

$$\frac{1}{2\pi}\int_0^\pi dx\, h(x) = \frac{1}{2\pi}\exp(t_4)\int_0^\pi dx\, \exp\left(-\frac{2\,t_4}{1-\cos(x)}\right) = \frac{1}{2}\exp(t_4)\,\mathrm{erfc}(\sqrt{t_4}) = \frac{1}{2}\,\mathrm{erfcx}(\sqrt{t_4}), \tag{87}$$

where the last function is an available MATLAB routine. The function erfcx($x$) is well behaved for all positive real $x$ and decays as $1/(\sqrt{\pi}\, x)$ for large $x$.

The pertinent integral in equations (80) through (82) can now be expressed as

$$\frac{1}{2\pi}\int_0^\pi dx\, g(x) = \frac{1}{2\pi}\int_0^\pi dx\, [g(x)-h(x)] + \frac{1}{2}\,\mathrm{erfcx}(\sqrt{t_4}), \tag{88}$$

leaving the major numerical problem as

$$\frac{1}{2\pi}\int_0^\pi dx\, w(x), \quad w(x) \equiv g(x)-h(x) = \exp\left(-t_4 \frac{1+\cos(x)}{1-t_3\cos(x)}\right) - \exp\left(-t_4 \frac{1+\cos(x)}{1-\cos(x)}\right). \tag{89}$$

The difference function $w(x)$ has the special values

$$w(0) = \exp(-2\,a\,b), \;\; w(\pi/2) = 0, \;\; w(\pi) = 0, \tag{90}$$

indicating that a major portion of function $g(x)$ has been accounted for, by means of function $h(x)$ defined in equation (86). Integral (89) is also a good candidate for MATLAB's quad or quad$\ell$ routines.

## EVALUATION OF $Q(a,b)$ FOR COMPLEX ARGUMENTS

The integrand in equation (85) can be extended to the complex $z$-plane by analytic continuation:

$$g(z) = \exp\left(-t_4 \frac{1+\cos(z)}{1-t_3\cos(z)}\right). \tag{91}$$

The function $g(z)$ has an essential singularity (ES) where

$$\cos(z_o) = 1/t_3, \quad z_o = \text{acos}(1/t_3), \quad t_3 = \frac{2ab}{a^2 + b^2}. \tag{92}$$

However, this complex value $z_o$ is only the principal value location of the ESs of $g(z)$. The totality of ESs of $g(z)$ are located at

$$z_o + 2\pi n \text{ and } -z_o + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{93}$$

Principal value $z_o$ has a real part that always lies in the interval $[0,\pi]$. Therefore, the only ESs that can affect the integration over $[0,\pi]$ in equation (82) are those at

$$\pm z_o \text{ and } \pm z_o + 2\pi. \tag{94}$$

If input arguments $a$ and $b$ are real, positive, and unequal, then $t_3$ is real and less than 1, and

$$z_o = \text{acos}(1/t_3) = i \,\text{acosh}(1/t_3), \tag{95}$$

which lies on the imaginary axis of the z-plane. Another ES lies at $-z_o$. Therefore, the path of integration in equation (82) starts out *perpendicular* to the line between the two ESs on the imaginary axis. This path is the best way of avoiding the neighborhoods of the two closest ESs, and yet begin the integration at $z = 0$. The two other ESs in equation (94) are located above and below $2\pi$ in the $z$-plane and do not significantly affect the integrand in the interval $[0,\pi]$.

On the other hand, when either input argument $a$ and $b$ is complex, principal value location $z_o$ in equation (92) can have a real part that approaches $\pi$. Then, the real part of $2\pi - z_o$ also approaches $\pi$, thereby possibly leading to two ESs close to the end of the path of integration at $z = \pi$. To try to stay away from the immediate neighborhoods of these ESs, the following modification of the path of integration in the $z$-plane is suggested. First, draw a line $L_1$ between $z_o$ and $-z_o$. Then, start the integration from $z = 0$ along a line $P_1$ perpendicular to $L_1$. Second, draw a line $L_2$ between $z_o$ and $2\pi - z_o$. Finish the integration at $z = \pi$ along a line $P_2$ perpendicular to $L_2$. This approach attempts to avoid the neighborhoods of the ESs, wherever they may be located. The lengths of the two perpendicular lines $P_1$ and $P_2$ could be taken as $\pi/2$; this guarantees that the projections of the two perpendicular lines $P_1$ and $P_2$ on the $x$-axis will not extend beyond each other, yet allows the path of integration to get a sufficient distance away from the ESs. Finally, join the ends of the two perpendicular lines $P_1$ and $P_2$ with another line, thereby completing the modified contour between $z = 0$ and $z = \pi$.

This procedure has been found to be adequate for some complex pairs of arguments $a, b$, but not for all arguments. When the magnitudes of arguments $a,b$ get large, some pairs of arguments cause large variations in the exponential functions involved, and all significance can be lost.

## ALTERNATIVE EVALUATION OF THE $Q_M(a,b)$ FUNCTION

The $Q_M(a,b)$ function can be expressed in terms of the $Q(a,b)$ function according to (reference 4, equation (3)):

$$Q_M(a,b) = Q(a,b) + \exp\left(-\frac{(a-b)^2}{2}\right)\sum_{k=1}^{M-1}\left(\frac{b}{a}\right)^k \exp(-ab)\, I_k(ab). \tag{96}$$

Again, the same underflow factor as in equations (76) through (81) appears in the additive term; additionally, the well-behaved combination $\exp(-x)\, I_k(x)$ is encountered. Therefore, a simple modification to equations (81) and (82) enables evaluation of $Q_M(a,b)$ via equation (96). The apparent danger of small $a$ in the $b/a$ term is partially compensated for, by the fact that

$$\frac{I_k(x)}{x^k} \sim \frac{1}{2^k\, k!} \quad \text{as } x \to 0. \tag{97}$$

However, for complex $a,b$ if the sequence $\{\exp(-ab)\, I_k(ab)\}$ has been determined by means of a recursion using subtraction, form (96) will lose significance for small $a \neq 0$. Then, it will be necessary to employ a power series expansion of

$$\sum_{k=1}^{M-1}\left(\frac{b}{a}\right)^k I_k(ab) \tag{98}$$

about $a = 0$. The case of $a = 0$ must be taken care of separately.

A routine for the evaluation of the four quantities

$$Q_M(a,b),\ q_M(a,b) = 1 - Q_M(a,b),\ \log(Q_M(a,b)),\ \log(q_M(a,b)) \tag{99}$$

was written and made available with the call

$$[Q, q, Q\log, q\log] = Qq\log M(M,a,b). \tag{100}$$

At least one of the four quantities in equation (99) retains significance for all arguments. This routine can be extended to complex arguments $a,b$, but is not trustworthy for all complex arguments.

## EVALUATION OF THE $Q_M(a,b)$ FUNCTION FOR COMPLEX ARGUMENTS BY MEANS OF INTEGRATION ON THE REAL AXIS

The $Q_M(a,b)$ function was defined in equation (1) as

$$Q_M(a,b) = \int_b^\infty dx\, x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2+a^2}{2}\right) I_{M-1}(a\,x). \tag{101}$$

The integrand is analytic for all finite complex $a$, thereby leading to the $Q_M(a,b)$ function being analytic in both variables $a,b$ for all finite complex values. The complementary $Q_M$ function was defined as

$$q_M(a,b) = 1 - Q_M(a,b) = \int_0^b dx\, x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2+a^2}{2}\right) I_{M-1}(a\,x). \tag{102}$$

By expanding the Bessel function in equation (101) in a power series and integrating term by term, an alternative expression is obtained (see reference 4, equation (8)):

$$Q_M(a,b) = \exp\left(-\frac{a^2+b^2}{2}\right) \sum_{k=0}^{\infty} \frac{(a^2/2)^k}{k!} \sum_{n=0}^{k+M-1} \frac{(b^2/2)^n}{n!}. \tag{103}$$

This form allows for a useful observation, namely, that the $Q_M(a,b)$ function is even in both arguments. That is,

$$Q_M(a,b) = Q_M(-a,b) = Q_M(a,-b) = Q_M(-a,-b), \tag{104}$$

even when arguments $a$ and $b$ are complex. Advantage will be taken of this fact later. However, direct use of expansion (103) for numerical evaluations leads to underflow and overflow for moderately large values of arguments $a$ and $b$, and to cancellation and loss of significance for complex $a,b$. Accordingly, an alternative representation of the $Q_M(a,b)$ function is sought that will allow its accurate evaluation for large complex arguments $a,b$.

By repeated integration by parts on equation (101) (see reference 4, equation (4)), another representation is achieved:

$$Q_M(a,b) = \exp(-A-B) \sum_{k=1-M}^{\infty} t^k\, I_k(u), \quad \text{where } A = \frac{a^2}{2},\ B = \frac{b^2}{2},\ t = \frac{a}{b},\ u = a\,b. \tag{105}$$

This expression can be developed as follows:

$$Q_M(a,b) = \exp(-A-B)\, t^{1-M} \sum_{k=1-M}^{\infty} t^{k-1+M}\, I_k(u) = \exp(-A-B)\, t^{1-M} \sum_{j=0}^{\infty} t^j\, I_{j+1-M}(u)$$
$$= \exp(-A-B)\, t^{1-M}\, \frac{1}{\pi} \int_0^{\pi} dx\, \exp[u\cos(x)] \sum_{j=0}^{\infty} t^j \cos[(j+1-M)x]. \tag{106}$$

The sum on $j$ can developed into a more useful form as

$$\frac{1}{2} \sum_{j=0}^{\infty} t^j\, \{\exp[i(j+1-M)x] + \exp[-i(j+1-M)x]\}$$
$$= \frac{1}{2}\exp[-ix(M-1)] \sum_{j=0}^{\infty} [t\exp(ix)]^j + \frac{1}{2}\exp[ix(M-1)] \sum_{j=0}^{\infty} [t\exp(-ix)]^j \tag{107}$$
$$= \frac{1}{2}\frac{\exp[-ix(M-1)]}{1-t\exp(ix)} + \frac{1}{2}\frac{\exp[ix(M-1)]}{1-t\exp(-ix)} \quad \text{for } |t|<1, \text{ that is, } |a|<|b|.$$

Substitution in equation (106) yields

$$Q_M(a,b) = \exp(-A-B)\, t^{1-M}\, \frac{1}{2\pi} \int_0^{\pi} dx\, \exp[u\cos(x)] \left( \frac{\exp[ix(M-1)]}{1-t\exp(-ix)} + \frac{\exp[-ix(M-1)]}{1-t\exp(ix)} \right)$$
$$= \exp(-A-B)\, t^{1-M}\, \frac{1}{\pi} \int_0^{\pi} dx\, \exp[u\cos(x)] \frac{\cos[(M-1)x] - t\cos(Mx)}{1-2t\cos(x)+t^2} \tag{108}$$
$$= -\exp\left(-\frac{(a-b)^2}{2}\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_0^{\pi} dx\, \exp[-ab(1-\cos(x))] \frac{\cos(Mx) - \frac{b}{a}\cos((M-1)x)}{\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right) - \cos(x)}$$

for $|a|<|b|$, where the definitions in equation (105) have been utilized.

In an exactly analogous fashion, the expansion for complementary function $q_M(a,b)$ in reference 4, equation (5), can be developed into an integral representation. The end result is identical to that in equation (108), except for the minus sign out front and the alternative restriction now that $|b|<|a|$. Therefore, the situation can be summarized as follows: define integral

$$I = \exp\left(-\frac{(a-b)^2}{2}\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_0^{\pi} dx\, \exp[-ab(1-\cos(x))] \frac{\cos(Mx) - \frac{b}{a}\cos((M-1)x)}{\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right) - \cos(x)}. \tag{109}$$

Then (see also, reference 9, equations (C-26) and (C-27))

$$q_M(a,b) = +I \text{ if } |b|<|a|,$$
$$Q_M(a,b) = -I \text{ if } |a|<|b|. \tag{110}$$

The main issue, then, is the evaluation of the integral $I$ in equation (109) for complex $a,b$. Extend the integrand of equation (109) into the complex $z$-plane by analytic continuation and define numerator

$$N(z) = \exp[-a\,b\,(1-\cos(z))]\left(\cos(M\,z) - \frac{b}{a}\cos((M-1)\,z)\right). \tag{111}$$

This function is entire in $z$; that is, $N(z)$ has no singularities anywhere in the finite $z$-plane. Also, the analytic continuation of the denominator is

$$D(z) = \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right) - \cos(z). \tag{112}$$

This function $D(z)$ has simple zeros at $z = z_o$, where

$$\cos(z_o) = \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right), \text{ or } \exp(\mathrm{i}\,z_o) = t, \frac{1}{t} = \frac{a}{b}, \frac{b}{a}, \tag{113}$$

as seen from the first line of equation (108). Therefore,

$$i\,z_o = i\,2\pi\,n \pm \log\left(\frac{b}{a}\right) = i\,2\pi\,n \pm \left[\log\left|\frac{b}{a}\right| + i\text{ angle}\left(\frac{b}{a}\right)\right], \tag{114}$$

where $n$ is an arbitrary integer, positive or negative. Here, functions log($z$) and angle($z$) are the principal value functions. Finally, the locations of the zeros of denominator $D(z)$ are at

$$z_o = 2\pi\,n \pm \left[\text{angle}\left(\frac{b}{a}\right) - i\,\log\left|\frac{b}{a}\right|\right]. \tag{115}$$

One of these zero locations always has a real part that lies in $[0,\pi]$, which is the region of integration for $I$ in equation (109). The imaginary part of this zero location can lie above or below the $x$-axis, or it can lie right on the $x$-axis when $|a| = |b|$.

In fact, every strip, $[n\pi, (n+1)\pi]$, of the $z$-plane contains a zero of denominator $D(z)$. Furthermore, some pairs of these zeros can lie very close to each other. For example, if $z_o$ is near 0, then the zero at $-z_o$ is also near 0. Or if $z_o$ is near $\pi$, then $2\pi - z_o$ is also near $\pi$. Since the zeros of $D(z)$ are the simple poles of the integrand $N(z)/D(z)$ of equation (109), direct numerical evaluation of $I$ by means of equation (109) will run into difficulties when these zeros lie close to the real $x$-axis, that is, $|a| \approx |b|$. However, equation (109) is imminently useable when $|b/a|$ is sufficiently different from unity, because all the poles of $N(z)/D(z)$ are sufficiently removed from the neighborhood of the path of integration over $[0,\pi]$. Furthermore, equation (109) is much more attractive than equations (101) or (102) for two reasons: no Bessel functions need to be evaluated and the range of integration is finite. Also, only one exp function, and no more than three cos functions, need to be evaluated, all on the real $x$-axis. In fact, for $M = 1$, only one cos function is involved; for $M = 2$, only two cos function evaluations are required.

At first sight, it might appear that the contour of integration in equation (109) could be moved away from the pole locations of $N(z)/D(z)$, by moving the contour upward or downward, into the half-plane away from the nearest pole. This procedure has two drawbacks. First, the cos functions in equation (109) develop exponential behavior because

$$\cos(x+i\,y) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) \tag{116}$$

for nonzero $y$, thereby leading to large oscillating behavior of the integrand with $x$. Second, when the poles are close to 0 or $\pi$, it is not possible to get away from these poles because the path of integration must start and stop at these points.

To handle the case where $|a| \approx |b|$, or the poles are close together, the singularities of the integrand, $N(z)/D(z)$, in equation (109) will be subtracted out and integrated *analytically*. To accomplish this task, observe from equation (113) that

$$\exp(i\,k\,z_o) = \left(\frac{a}{b}\right)^k, \left(\frac{b}{a}\right)^k; \quad \cos(k\,z_o) = \frac{1}{2}\left[\left(\frac{a}{b}\right)^k + \left(\frac{b}{a}\right)^k\right]. \tag{117}$$

Therefore,

$$\cos(M\,z_o) - \frac{b}{a}\cos((M-1)\,z_o) = \frac{a^{M-2}\,(a^2-b^2)}{2\,b^M}. \tag{118}$$

Reference to equations (111), (113), and (118) then yields

$$N(z_o) = \exp\left[-a\,b\left(1-\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)\right)\right]\frac{a^{M-2}\,(a^2-b^2)}{2\,b^M} = \exp\left(\frac{1}{2}\,(a-b)^2\right)\frac{a^{M-2}\,(a^2-b^2)}{2\,b^M}. \tag{119}$$

Now, express the integrand of equation (109) as

$$\frac{N(x)}{D(x)} = \frac{N(x)-N(z_o)}{D(x)} + \frac{N(z_o)}{D(x)}. \tag{120}$$

The integral of the last term in equation (120) is

$$\int_0^\pi dx \frac{N(z_o)}{D(x)} = \int_0^\pi dx \frac{N(z_o)}{\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)-\cos(x)} = N(z_o)\frac{2\pi ab}{a^2-b^2}\, p \quad \text{for } b \neq \pm a, \tag{121}$$

where polarity

$$p = \begin{cases} -1 & \text{if } |a| < |b| \\ +1 & \text{if } |b| < |a| \end{cases}. \tag{122}$$

Here, the integral result is used

$$\int_0^\pi dx \frac{1}{\alpha - \cos(x)} = \frac{\pi}{\sqrt{\alpha+1}\,\sqrt{\alpha-1}} \quad \text{if } \alpha \notin [-1,1], \tag{123}$$

where both square roots are principal value. (The branch line of the product of square roots is along the real axis of the complex $\alpha$ plane, between $\alpha = -1$ and $\alpha = 1$. The denominator of equation (123) cannot be replaced by principal-value square root $\sqrt{\alpha^2 - 1}$.)

Combining the results in equations (119) and (121) yields

$$\int_0^\pi dx \frac{N(z_o)}{D(x)} = \pi \exp\left(\frac{1}{2}(a-b)^2\right)\left(\frac{a}{b}\right)^{M-1} p. \tag{124}$$

Finally, use of this result and integrand breakdown (120) yields, for the second component of equation (109), the very simple result

$$I_2 = \frac{1}{2}\, p. \tag{125}$$

The first component of equation (109) is given by

$$I_1 = \exp\left(-\frac{(a-b)^2}{2}\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi}\int_0^\pi dx \frac{N(x)-N(z_o)}{D(x)}. \tag{126}$$

But the analytic continuation of this integrand has *no singularities* anywhere in the finite $z$-plane. The subtraction procedure in equation (120) has accounted for *all the poles* of $N(z)/D(z)$ in the subtracted term $N(z_o)/D(z)$. Therefore, numerical evaluation of the integral in equation (126) can proceed without fear of encountering any singularities on or near the path of integration.

At this point, equations (109), (125), (126), and (122) yield

$$I = I_1 + I_2 = I_1 + \frac{1}{2}p = \left\{ \begin{matrix} I_1 + \frac{1}{2} & \text{if } |b| < |a| \\ I_1 - \frac{1}{2} & \text{if } |a| < |b| \end{matrix} \right\}. \tag{127}$$

When this information is coupled with equation (110), there follows the major simplification

$$Q_M(a,b) = \frac{1}{2} - I_1, \quad q_M(a,b) = \frac{1}{2} + I_1, \tag{128}$$

which holds for *all* relative sizes of $|a|$ and $|b|$. The only cases where the integral for $I_1$ in equation (126) cannot be used is when $a = 0$ or $b = 0$. However, both of these results are already known in closed form, namely,

$$Q_M(0,b) = \exp\left(-\frac{b^2}{2}\right) \sum_{m=0}^{M-1} \frac{1}{m!}\left(\frac{b^2}{2}\right)^m, \quad Q_M(a,0) = 1. \tag{129}$$

The quantities needed for determination of the integrand of equation (126) are given in equations (111), (112), and (119). The integral in equation (126) is to be used in conjunction with equation (128) when $|b/a|$ is fairly near unity. Otherwise, the earlier form in equation (109) is to be used directly. The reason that equations (126) and (128) are not used all the time is that form (109) retains full significance even when $Q_M(a,b)$ and $q_M(a,b)$ become extremely small or large, whereas form (128) retains full significance only when $Q_M(a,b)$ and $q_M(a,b)$ are in the neighborhood of 0.5. When form (109) would tend to underflow or overflow, the logarithm of $I$ can be easily computed because the leading exponential factor is the major cause of that problem; thus, $\log(Q_M)$ and $\log(q_M)$ can be output in these cases, in addition to $Q_M$ and $q_M$.

At $x = 0$ in equations (109) and (126), the $\exp[-ab(1-\cos(x))]$ term is 1. However, as $x$ increases from 0 toward $\pi$, this term can grow very fast in magnitude, reaching value $\exp(-2ab)$. To avoid this growth, the polarity of product $ab$ can be switched, if necessary, so that

$$\text{real}(a\,b) > 0 \tag{130}$$

before equations (109) or (126) are numerically evaluated. This switch is permissible according to the results in equation (104). The remaining quantities in the two different integrands involve cos functions evaluated along the real axis. And the dangers of a small denominator $D(x)$ have been avoided by employing equation (126) when $|a| \approx |b|$. The only difficult case not covered by this switch is when $|\text{imag}(a\,b)|$ is large, because the $\exp[-ab(1-\cos(x))]$ term will then oscillate very quickly with $x$, even as its magnitude decreases. This will necessitate numerous integrand evaluations in order to attain an accurate integral result. In the very special case where arguments $a$ and $b$ are complex, but their product is *real*, this oscillatory problem will not arise at all; however, switch (130) is still recommended. The oscillations of the remaining cos functions in the integrand of equations (109) and (126) are tolerable for moderate values of $M$.

Another case that requires special treatment is when $b = a$ or $b = -a$. Reference to equation (104) and the closed-form result in reference 4, equation (7), yields

$$Q_M(a, \pm a) = \frac{1}{2} + \exp(-a^2)\left(\frac{1}{2} I_0(a^2) + \sum_{m=1}^{M-1} I_m(a^2)\right). \tag{131}$$

Instead of subtracting out the constant $N(z_o)$ in the numerator, as done in equation (120), it is possible to subtract other factors instead. If so, a useful integral result, relative to equation (109), is

$$\frac{1}{\pi}\int_0^{\pi} dx \, \frac{\cos(M\,x) - r\cos((M-1)\,x)}{\frac{1}{2}\left(r + \frac{1}{r}\right) - \cos(x)} = \begin{Bmatrix} -2/r^{M-1} & \text{for } |r| > 1 \\ -1 & \text{for } |r| = 1 \\ 0 & \text{for } |r| < 1 \end{Bmatrix}. \tag{132}$$

A MATLAB routine was written to incorporate the above developments; it is called according to

$$[Q, q, Q\log, q\log] = Qq\log(M, a, b)\,. \tag{133}$$

This routine outputs the four quantities $Q_M$, $q_M$, $\log(Q_M)$, and $\log(q_M)$. At least one of these quantities always retains full significance. However, cases arise where the two quantities that have to be subtracted from each other are much larger than the difference, thereby losing all significance in the end result.

## STEEPEST DESCENT EVALUATION OF INTEGRALS

The basic problem of interest here is to evaluate a contour integral

$$I = \int_C dz\, \Psi(z), \tag{134}$$

where $C$ is a Bromwich contour extending from $-i\infty$ to $+i\infty$. It is presumed that contour $C$ can be moved so as to pass through a (simple) SP of the integrand $\Psi(z)$. Define

$$\psi(z) = \log[\Psi(z)]; \quad \text{then } \Psi(z) = \exp[\psi(z)] = \exp[\psi_r(z)] \exp[i\, \psi_i(z)], \tag{135}$$

where subscripts $r$ and $i$ denote the real and imaginary parts, respectively. The locations $\{z_s\}$ of the SPs of the integrand $\Psi(z)$ are located where

$$\psi'(z_s) = 0. \tag{136}$$

Let $C_s$ denote a steepest descent (SD) contour passing through an SP at $z_s$. Then, a very important property of analytic functions on a path of SD is that

$$\psi_i(z) = \psi_i(z_s) \text{ for } z \text{ on } C_s. \tag{137}$$

Thus, from equation (135), the angle $\psi_i(z)$ of integrand $\Psi(z)$ on contour $C_s$ remains equal to its value at SP location $z_s$. Equation (137) is the key to finding the SD contours out of an SP at $z_s$.

Define difference function

$$\phi(z) = \psi(z) - \psi(z_s) \text{ for all } z. \tag{138}$$

Then,

$$\phi_i(z) = \psi_i(z) - \psi_i(z_s) = 0 \text{ for } z \text{ on } C_s. \tag{139}$$

An alternative way of stating this property is

$$\phi(z) \text{ is real for } z \text{ on } C_s, \text{ and } \phi(z_s) = 0. \tag{140}$$

Then, using equation (138), the integrand $\Psi(z)$ in equation (135) can be written as

$$\Psi(z) = \exp[\psi(z_s)] \exp[\phi(z)]. \tag{141}$$

Substitution in equation (134) yields

$$I = \int\limits_{C} dz\ \Psi(z) = \int\limits_{C_s} dz\ \Psi(z) = \exp[\psi(z_s)] \int\limits_{C_s} dz\ \exp[\phi(z)]. \tag{142}$$

On SD contour $C_s$, *real* function $\phi(z)$ decreases monotonically from the value 0 as $z$ moves away from SP location $z_s$, in either direction. Although integrand $\exp[\phi(z)]$ is real on contour $C_s$, the integral on $z$ is complex because $dz$ is complex. The integrand of equation (142) at $z = z_s$ is

$$\exp[\phi(z_s)] = \exp[0] = 1, \tag{143}$$

which is its maximum value anywhere on contour $C_s$.

The additional factor of $\exp[\psi(z_s)]$ in equations (141) and (142) is a complex constant and can be ignored during the evaluation of the contour integral in equation (142). However, this factor may cause underflow or overflow in result $I$. In such cases, the logarithm of $I$ can be computed according to

$$\log(I) = \psi(z_s) + \log\left( \int\limits_{C_s} dz\ \exp[\phi(z)] \right). \tag{144}$$

Although the log of the integral will produce the standard principal value logarithm, the remaining term may cause the imaginary part of $\log(I)$ to exceed the $(-\pi, \pi]$ range. In this case, the imaginary part of $\log(I)$ can be subject to a mod operation, which will return it to the range $(-\pi, \pi]$, if desired. Of course, if and when $I$ is finally computed by taking the exp of $\log(I)$, this modification will be irrelevant, since $\exp(i 2\pi n) = 1$ for $n$ integer.

The key property for determining SD contour $C_s$ out of SP location $z_s$ is, from equation (139),

$$\phi_i(z) = 0 \quad \text{for } z \text{ on } C_s. \tag{145}$$

To get started on contour $C_s$, consider the neighborhood of point $z_s$. Near this SP location $z_s$, the following approximation is very useful:

$$\phi(z) = \psi(z) - \psi(z_s) \cong \frac{1}{2} \psi''(z_s)\,(z - z_s)^2, \tag{146}$$

since $\psi'(z_s) = 0$, and presuming $\psi''(z_s) \neq 0$. For complex point $z_1$ (near $z_s$) to be on contour $C_s$, one must have (in keeping with equation (145)),

$$\frac{1}{2}\psi''(z_s)\,(z_1 - z_s)^2 \cong -f_s,\; f_s \text{ real and positive.} \tag{147}$$

That is,

$$z_1 \cong z_s \pm i\sqrt{\frac{2 f_s}{\psi''(z_s)}}. \tag{148}$$

These two values correspond to the two directions of the SD contours out of the (simple) SP at $z_s$. Then, from equations (146) and (147),

$$\phi(z_1) \cong -f_s, \text{ and } \exp[\phi(z_1)] \cong \exp(-f_s). \tag{149}$$

For example, if

$$f_s = 0.25, \text{ then } \exp[\phi(z_1)] \cong \exp(-0.25) = 0.78, \tag{150}$$

versus the SP value of $\exp[\phi(z_s)] = 1$. Equation (148) can be used to approximately find the first point(s) on $C_s$ away from SP location $z_s$, using $f_s$ of the order of 0.25.

The point(s) $z_r$ given by the right-hand side of equation (148) will not be exactly on SD contour $C_s$. Therefore, a local search in the neighborhood of $z_r$ will be necessary to locate the point $z_1$, where $\phi_i(z_1) = 0$. One possible procedure is to let the points (for the $+i$ alternative in equation (148))

$$z_\pm \equiv z_s + i\sqrt{\frac{2 f_s}{\psi''(z_s)}}\exp(\pm i\theta), \quad \theta \sim \frac{\pi}{10} = 18 \text{ degrees,} \tag{151}$$

which, hopefully, lie on opposite sides of contour $C_s$, and compute

$$\phi(z_-) \text{ and } \phi(z_+). \tag{152}$$

Then, since $\phi_i(z) = 0$ on $C_s$, solve for the linearly interpolated value of the zero crossing as

$$\theta_1 = \frac{\phi_i(z_-) + \phi_i(z_+)}{\phi_i(z_-) - \phi_i(z_+)}\,\theta. \tag{153}$$

Finally, take the first point away from SP location $z_s$ as

$$z_1 = z_s + i\sqrt{\frac{2f_s}{\psi''(z_s)}}\exp(i\theta_1). \tag{154}$$

Then, $\phi_i(z_1) \cong 0$ is expected, although not exactly. A similar procedure is used for the $-i$ alternative in equation (148), which corresponds to the opposite direction of SD out of the SP at $z_s$.

Once the movement away from the SP location has taken place, a different procedure is in order. At two close points $z_{n-1}, z_n$ on $C_s$,

$$\phi(z_n) = \phi(z_{n-1} + [z_n - z_{n-1}]) \cong \phi(z_{n-1}) + \phi'(z_{n-1})\,(z_n - z_{n-1}). \tag{155}$$

Presuming $\phi(z_{n-1})$ is real, then to keep $\phi(z_n)$ real, one must take

$$\phi'(z_{n-1})\,(z_n - z_{n-1}) \cong -f, \quad f \text{ real and positive.} \tag{156}$$

That is,

$$z_n \cong z_{n-1} - \frac{f}{\phi'(z_{n-1})} = z_{n-1} - \frac{f}{\psi'(z_{n-1})}. \tag{157}$$

Then, the new integrand value is

$$\exp[\phi(z_n)] \cong \exp[\phi(z_{n-1}) - f] = \exp[\phi(z_{n-1})]\exp(-f). \tag{158}$$

For example, if $f = 0.25$, the relative amplitude of the integrand at the two close points on $C_s$ is down by $\exp(-0.25) = 0.78$. Equation (157) cannot be used with the SP location $z_s$ as a starting point because $\psi'(z_s) = 0$, and equation (155) is not adequate there. An extension of equations (155) through (157) to second order is given in equations (166) and (167).

As point $z_n$ moves away from SP location $z_s$, $\psi'(z_n)$ possibly will become small, suggesting large increments $z_n - z_{n-1}$ according to equation (157). To prevent this, reconsider equation (148). The magnitude of the first increment away from the SP at $z_s$ is

$$|z_1 - z_s| \cong \sqrt{\frac{2f_s}{|\psi''(z_s)|}} \equiv \Delta_s. \tag{159}$$

Then, a possible alternative to equation (157) is

$$z_n = z_{n-1} - \Delta_s \frac{|\psi'(z_{n-1})|}{\psi'(z_{n-1})} = z_{n-1} - \Delta_s \frac{\psi'(z_{n-1})^*}{|\psi'(z_{n-1})|}, \tag{160}$$

which retains the direction information of equation (157) and does not allow large increments to develop. In general, one could use the rule

$$z_n = z_{n-1} - g(z_{n-1}), \tag{161}$$

where $|g(z_{n-1})|$ has the smaller magnitude of the two update terms in equations (157) and (160).

Since new point $z_n$ will not lie exactly on SD contour $C_s$, let

$$z_{\pm} \equiv z_{n-1} - g(z_{n-1}) \exp(\pm i\theta), \quad \theta \sim \frac{\pi}{10} = 18 \text{ degrees}. \tag{162}$$

Compute $\phi(z_{\pm})$ and interpolate as in equations (152) and (153). Finally, take

$$z_n = z_{n-1} - g(z_{n-1}) \exp(i\theta_1). \tag{163}$$

Again, $\phi_i(z_n) \cong 0$ is expected, although not exactly. This procedure requires two $\phi(z)$ calculations at each stage (at $z_{\pm}$) instead of one. However, it *continually corrects* itself by maintaining $\phi_i(z_n) \cong 0$ for each new contour point $z_n$. This recursion procedure is repeated until the real part of $\phi(z)$ is sufficiently negative that the integrand, $\exp[\phi(z)]$, in equation (142) is negligible or below a specified tolerance.

Once the SD paths out of an SP location have been determined, that is, the sequence of values $\{z_n\}$ on both sides of the SP, a numerical integration procedure is in order. The sequence of points $\{z_n\}$ will be close to the path of SD, but they will not lie directly on it. Nevertheless, *analytic* integration along the piecewise linear path afforded by locations $\{z_n\}$ would yield the exact same result for the integral as for the SD path, according to Cauchy's theorem. However, since an analytic integration is not possible in general, some numerical integration rule must be employed on the set of complex locations $\{z_n\}$ and the corresponding complex function values, $\{\exp[\phi(z_n)]\}$. For example, the parabolic interpolation and integration procedure for three adjacent complex locations $z_1, z_2, z_3$ is given by

$$z_{21} = z_2 - z_1, \; z_{32} = z_3 - z_2, \; z_{31} = z_3 - z_1,$$

$$r_1 = \frac{z_{31}}{z_{21}}(2z_{21} - z_{32}), \; r_2 = \frac{z_{31}^3}{z_{21}\, z_{32}}, \; r_3 = \frac{z_{31}}{z_{32}}(2z_{32} - z_{21}), \tag{164}$$

$$\text{sum} = \frac{1}{6}[r_1\, f(z_1) + r_2\, f(z_2) + r_3\, f(z_3)],$$

where $f(z)$ is the function to be integrated. When the three points $z_1, z_2, z_3$ all lie on a straight line and are equally spaced, that is,

$$z_{21} = h, \; z_{32} = h, \; z_{31} = 2h, \quad h \text{ complex}, \tag{165}$$

then $r_1 = 2h, r_2 = 8h, r_3 = 2h,$ and equation (164) reduces to Simpson's rule with complex function samples. However, the sequence $\{z_n\}$ yielded by recursion (163) will generally lie on a curved series of points near $C_s$ in the $z$-plane, necessitating the use of the general rule in equation (164).

Since the spacing between sample points $\{z_n\}$ is not zero, the numerical procedure employing equation (164) will not give an exact result for the desired integral. However, by sampling halfway between the existing points and evaluating the integrand at these intermediate points, re-evaluation of the sum by means of equation (164) (at twice as many points) will improve the accuracy of the approximation. There is no need to duplicate the function evaluations that have already taken place; only the function values at the halfway points need to be evaluated. This halving procedure can be repeated until two consecutive approximations for the integral of interest differ by less than some user-specified tolerance. The amount of storage is not excessive, although the number of stored locations and function values doubles with each stage of this procedure.

According to equations (135) and (138), $\phi(z)$ is the difference of two logarithms. During the tracking of the SD contour according to equations (162) and (163), it is possible that the branch line of the principal value logarithm log($z$) may be crossed. This will yield an undesired jump of $\pm 2\pi$ in the imaginary part of the logarithm. This jump would grossly affect the interpolation procedure in equations (152) and (153), because $\phi_i(z)$ *must be maintained* near zero on contour $C_s$. Since the $z$ increments along contour $C_s$ are small, according to equation (163), the simplest way to circumvent this discontinuity problem is to always force function $\phi(z)$ to return an imaginary part in the range $(-\pi, \pi]$. If function $\phi(z)$ contains other principal value functions, such as sqrt($z$), this same correction must be incorporated. It is not necessary to modify function $\psi(z)$, except possibly as noted under equation (144).

If the expansion of $\phi(z)$ is used to second-order, instead of the first-order expansion in equation (155), that is,

$$\phi(z_n) \cong \phi(z_{n-1}) + \phi'(z_{n-1})\,(z_n - z_{n-1}) + \frac{1}{2}\,\phi''(z_{n-1})\,(z_n - z_{n-1})^2, \tag{166}$$

there follows the alternative recursion

$$z_n = z_{n-1} - \frac{\phi'(z_{n-1})}{\phi''(z_{n-1})}\left[1 - \sqrt{1 - \frac{2f\,\phi''(z_{n-1})}{\phi'(z_{n-1})^2}}\right], \quad \phi''(z_{n-1}) \neq 0. \tag{167}$$

(As $\phi''(z_{n-1}) \to 0$, this rule approaches that in equation (157).)

When equation (136) is solved for the SP locations, namely,

$$\psi'(z_s) = 0, \tag{168}$$

there may be many solutions, depending on the nonlinearity of function $\psi'(z)$. Some of these SP locations may be useless; for example, those SD paths that lead into the locations of essential singularities or zeros may not be useful at all, or they may have to be augmented by an additional path leading over other SP locations on the way to the desired eventual destination in the $z$-plane. In other cases, it may be necessary to use more than one SP passage to piece together a complete path going over the desired extent in the $z$-plane. These alternatives can be discovered only by plotting the SD paths for the various SPs and choosing the desired one(s). Even then, some of the contour integral results for $I$ may lead to obvious results and not be useful. Some experimentation is required to find the right combination(s) for the particular problem at hand. However, as noted in equation (144), underflow and overflow can always be kept under control, and very accurate numerical results can be obtained.

## EXAMPLE OF NON-CENTRAL CHI-SQUARED VARIATE

The non-central chi-squared variate has the PDF (see equation (9))

$$p(u) = \alpha \left(\frac{\sqrt{2\alpha u}}{\beta}\right)^{M-1} \exp\left(-\alpha u - \frac{\beta^2}{2}\right) I_{M-1}(\beta\sqrt{2\alpha u}) \quad \text{for } u > 0, \tag{169}$$

where $\alpha$ and $\beta$ are real and positive. The corresponding MGF is

$$\mu(\lambda) = \int du \, \exp(\lambda u)\, p(u) = \left(1 - \frac{\lambda}{\alpha}\right)^{-M} \exp\left(\frac{\lambda}{\alpha - \lambda}\frac{\beta^2}{2}\right) \quad \text{for } \lambda_r = \text{real}(\lambda) < \alpha, \tag{170}$$

while the EDF is

$$e(v) = \int_v^\infty du \; p(u) = Q_M(\beta, \sqrt{2\alpha v}) \quad \text{for } v > 0. \tag{171}$$

The (analytic continuation of the) MGF $\mu(\lambda)$ has an ES at $\lambda = \alpha$.

Instead of trying to evaluate the EDF by means of integral (171), an alternative is available by using the inverse Laplace transform result

$$p(u) = \frac{1}{i\,2\pi} \int_{C_1} d\lambda \, \exp(-u\,\lambda)\, \mu(\lambda), \tag{172}$$

where Bromwich contour $C_1$ must initially be in the region of analyticity of the MGF, namely, to the left of the ES at $\lambda = \alpha$. Then, for $v > 0$, the EDF is given by

$$e(v) = \int_v^\infty du\, p(u) = \frac{1}{i\,2\pi} \int_{C_1} d\lambda\, \mu(\lambda) \int_v^\infty du\, \exp(-u\,\lambda) = \frac{1}{i\,2\pi} \int_{C_2} \frac{d\lambda}{\lambda} \mu(\lambda) \exp(-v\lambda), \tag{173}$$

where contour $C_2$ must now pass between the pole at $\lambda = 0$ and the ES at $\lambda = \alpha$. The additional restriction is required for the latter $u$ integral in equation (173) to converge, that is, $\lambda_r > 0$.

Now substitute equation (170) into equation (173), and equate the result to equation (171), thereby getting an explicit integral relation for the generalized $Q_M$ function as

$$Q_M(\beta, \sqrt{2\,\alpha\, v}) = \frac{1}{i\,2\pi} \int_{C_2} \frac{d\lambda}{\lambda} \exp(-v\,\lambda) \left(1 - \frac{\lambda}{\alpha}\right)^{-M} \exp\left(\frac{\lambda}{\alpha - \lambda} \frac{\beta^2}{2}\right). \tag{174}$$

Now make the substitutions

$$\alpha = \frac{b^2}{2\,v}, \quad \beta = a \quad (a \text{ and } b \text{ real and positive}) \tag{175}$$

to get the form

$$Q_M(a,b) = \frac{1}{i\,2\pi} \int_{C_2} \frac{d\lambda}{\lambda} \exp(-v\,\lambda) \left(1 - \frac{2\,v\,\lambda}{b^2}\right)^{-M} \exp\left(\frac{\lambda}{b^2/(2\,v) - \lambda} \frac{a^2}{2}\right). \tag{176}$$

Finally, let $\lambda = z\,\dfrac{b^2}{2\,v}$ and get

$$Q_M(a,b) = \frac{1}{i\,2\pi} \int_{C_3} \frac{dz}{z} \exp(-B\,z)\,(1-z)^{-M} \exp\left(\frac{A\,z}{1-z}\right), \tag{177}$$

where

$$A = a^2/2, \quad B = b^2/2, \tag{178}$$

and contour $C_3$ must pass between the pole at $z = 0$ and the ES at $z = 1$ of the integrand of equation (177). Under the identification

$$\Psi(z) = \frac{1}{z}\exp(-B\,z)\,(1-z)^{-M}\,\exp\!\left(\frac{A\,z}{1-z}\right), \tag{179}$$

the integral of interest takes the form

$$Q_M(a,b) = \frac{1}{i2\pi}I, \quad I = \int_{C_3} dz\,\Psi(z). \tag{180}$$

The residue of $\Psi(z)$ at $z = 0$ is obviously 1, as seen from equation (179); that is, Res(0) = 1. Since $B$ is real and positive, the contour $C_3$ can be moved far into the right-half $z$-plane, where the integrand $\Psi(z)$ tends to zero. The only singularity encountered in this movement is the ES at $z = 1$. Therefore, $I = -i2\pi\,\mathrm{Res}(1)$, where the minus sign is due to the negative (clockwise) sense of the encirclement of the ES at $z = 1$. Thus, equation (180) yields the very useful observation that

$$\mathrm{Res}(1) = -Q_M(a,b). \quad (\text{Also, } \mathrm{Res}(0) = 1.) \tag{181}$$

The situation can now be summarized as follows. Any contour $C$ that encircles just the pole at $z = 0$ (in a positive sense) will yield the value 1 for the integral

$$J \equiv \frac{1}{i2\pi}\int_C dz\,\Psi(z). \tag{182}$$

Also, any contour $C$ that encircles just the ES at $z = 1$ will yield the value $-Q_M(a,b)$ for $J$. Any contour $C$ that encircles both singularities will yield the value $1 - Q_M(a,b)$ for $J$. These observations will allow easier interpretation of the numerical results for the SPs and associated SD paths to be presented in the examples below.

According to equations (135) and (179), the function

$$\psi(z) = \log[\Psi(z)] = -B\,z + \frac{A\,z}{1-z} - \log(z) - M\log(1-z), \tag{183}$$

where $A$ and $B$ are given by equation (178). This function in equation (183) uses two principal value logarithms; therefore, the correction procedure on $\phi(z) = \psi(z) - \psi(z_s)$, outlined two paragraphs below equation (165), must be employed. Then,

$$\psi'(z) = -B + \frac{A}{(1-z)^2} - \frac{1}{z} + \frac{M}{1-z} = -\frac{B\,z^3 + (M+1-2B)\,z^2 + (B-A-2-M)\,z + 1}{z\,(1-z)^2},$$
$$\psi''(z) = \frac{2A}{(1-z)^3} + \frac{1}{z^2} + \frac{M}{(1-z)^2}. \tag{184}$$

Saddlepoint locations $\{z_s\}$ of integrand $\Psi(z)$ are determined by the three solutions of the cubic equation

$$B\,z_s^3 + (M+1-2B)\,z_s^2 + (B-A-2-M)\,z_s + 1 = 0. \tag{185}$$

Although this cubic equation has an analytic solution, it is not a simple one, even for low-order integers $M$. Rather, given numerical values of $A$, $B$, and $M$, the three SP locations $\{z_s\}$ can be determined numerically. It is then necessary to determine the SD contours leading out of each of these SP locations, and to determine which of these contours are useful, according to the discussion accompanying equation (182).

For $A$ and $B$ real and positive, it is seen from equation (179) that, on the real axis, $\Psi(z)$ is negative to the left of the origin and has a single maximum in that region; that is, its magnitude has a single *minimum*. In the region between $z = 0$ and $z = 1$, $\Psi(z)$ is positive and has a single minimum. To the right of $z = 1$, the magnitude of $\Psi(z)$ has a single *maximum*; observe that the value of the ES term rapidly approaches zero as $z \to 1+$. These three locations correspond to the SP locations for this case. However, the SD directions for the third SP are along the real axis of the $z$-plane and one path heads directly into the ES at $z = 1$, making this SP useless.

For the SP located between 0 and 1, on the other hand, the SD paths start out perpendicular to the real axis, and then both bend to the right, due to the $\exp(-B\,z)$ term. This is contour $C_3$ in equation (180). The only singularity encircled (in the negative sense) by this contour is that at $z = 1$, yielding, upon use of equation (181),

$$\frac{1}{i2\pi}\int_{C_3} dz\,\Psi(z) = -\text{Res}(1) = Q_M(a,b), \tag{186}$$

which is consistent with equations (180) and (182).

For the SP location to the left of the origin, the SD paths also start out perpendicular to the real axis, and then both bend to the right, again due to the $\exp(-B\,z)$ term. Call this contour $C_4$. Therefore, both singularities of $\Psi(z)$ are now encircled (in the negative sense), leading to a value for the integral of

$$\frac{1}{i2\pi}\int_{C_4} dz\,\Psi(z) = -\text{Res}(0) - \text{Res}(1) = -1 - [-Q_M(a,b)] = Q_M(a,b) - 1. \tag{187}$$

This is consistent with equation (182). Thus, contour $C_4$ through the leftmost SP location yields not $Q_M(a,b)$, but rather $Q_M(a,b)-1$. This can be a very useful observation when $Q_M(a,b)$ is very close to 1. The complementary function

$$q_M(a,b) = 1 - Q_M(a,b) \tag{188}$$

is the cumulative distribution and can be evaluated very accurately far out on its tail values by means of contour $C_4$, whereas its evaluation by the difference in equation (188), coupled with contour integral (186) on contour $C_3$, could have no significance at all. This example illustrates that some investigation is in order for the various SPs and their associated paths of SD, in order to find out which is the most useful path for the problem at hand.

## NUMERICAL EXAMPLES

A series of numerical examples of equation (179) is now presented that will further demonstrate the utility and futility of some of the various SPs. The first case is

$$M = 3,\ \ a = 1.1,\ \ b = 2.1;\ \ z_s = -1.2866,\ \ 0.30086,\ \ 1.1716. \tag{189}$$

This case bears out all the observations made above. The SP locations $\{z_s\}$ will be labeled 1,2,3 in the order presented here. Also, $Q_M(a,b)$ will be abbreviated by $Q$ in these examples.

The second case is

$$M = 3,\ \ a = 1.1,\ \ b = 21;\ \ z_s = -0.00461,\ \ 0.94054,\ \ 1.0459. \tag{190}$$

Since $b$ is much larger then $a$, there follows $Q = 3.001\text{e}-85 - \text{i}\,1.352\text{e}-101$, which was obtained by integration along the path passing through the second SP (at $z_s = 0.94054$). This path encircled the ES in the negative sense, thereby yielding equation (186) again. The path through the first SP encircled both singularities in the negative sense, and should yield $-\text{Res}(0) - \text{Res}(1) = -1 + Q$, which is virtually $-1$. The numerical integration confirmed this result; however, this is not a useful path or result due to the loss of significance. The third SP was useless, with an SD path leading into the ES.

The third case is

$$M = 3,\ \ a = 21,\ \ b = 1.1;\ \ z_s = -21.725,\ \ 0.00445,\ \ 17.109. \tag{191}$$

Since $a$ is much larger than $b$, there follows $-q = -1 + Q = -1.047\text{e}-91 - \text{i}\,3.198\text{e}-107$, which was obtained by integration along the path passing through the first SP. This path encircled both singularities in the negative sense, yielding $-\text{Res}(0) - \text{Res}(1) = -1 + \text{Q} = -\text{q}$, as expected. The path through the second SP encircled only the pole at the origin in a positive sense, yielding integral value Res(0)=1. This is correct but not useful. The third SP was useless.

The expression for $Q_M(a,b)$ in equations (177) and (178) is analytic in $a$ and $b$ for all $a$ and $b$. Therefore, by use of analytic continuation, the integral expression can be used for complex values of $a$ and $b$. Accordingly, the fourth case is

$$\begin{aligned} &M=3,\ a=1.1,\ b=2.1+i\,2; \\ &z_s=0.1425+i\,0.1240,\ \ 0.6374+i\,0.8624, 1.1738-i\,0.0362. \end{aligned} \tag{192}$$

Since $b$ is complex, the SP locations can now lie in the complex $z$-plane, not just on the real axis. The path through the first SP starts and ends in the fourth quadrant and encircles only the pole at the origin in a positive sense; the value of the integral is 1, which is not a useful result. The path through the second SP encircles both singularities in the negative sense, yielding integral value $Q-1=-q=2.1123-i\,4.6039$. The path through the third SP heads into the ES and is not useful.

The fifth case is

$$\begin{aligned} &M=3,\ a=1.1+i\,3,\ b=2.1/a; \\ &z_s=0.0495-i\,0.2181,\ \ 0.2505+i\,0.9875,\ \ 15.831-i\,12.742. \end{aligned} \tag{193}$$

Here, both $a$ and $b$ are complex, although their product is real. The path through the *third* SP encircles both singularities in a positive sense, yielding value $\mathrm{Res}(0)+\mathrm{Res}(1)=1-\mathrm{Q}=\mathrm{q}$, which is a useful result. The path through the first SP starts in the second quadrant and ends up heading into the ES at $z=1$; by itself, this is not useful. However, the path through the second SP comes out of the ES and heads off into the second quadrant. The *combination* of these two paths encircles the pole in the positive sense, but does *not* encircle the ES, though these two SD paths both enter into the ES at the same angle. Therefore, the sum of the two integrations obtained by using the combination path is equal to Res(0)=1. This is an interesting combination of SD paths, but it does not lead to a useful result.

The sixth case is

$$M=3,\ a=10,\ b=i\,10;\ \ z_s=0.009615,\ \ 1.0352\pm i\,1.0042. \tag{194}$$

The path through the first SP encircles the origin, yielding value Res(0)=1. As above, the combination of the two paths through the two remaining conjugate SPs encircles both singularities in the positive sense, yielding $\mathrm{Res}(0)+\mathrm{Res}(1)=1-\mathrm{Q}=\mathrm{q}=-0.026941$. The path through the upper SP stays in the upper-half $z$-plane, while the path through the lower SP stays in the lower-half $z$-plane.

The seventh case is

$$M = 3, \; a = 10 + i, \; b = 0.5 + i\,10;$$
$$z_s = 0.00965 - i\,0.00048, \; 0.8848 + i\,1.001, 1.1849 - i\,0.9926. \qquad (195)$$

Now, both parameters $a$ and $b$ are complex, and the SP locations have no symmetry properties. The path through the first SP encircles just the origin, yielding integral value 1 as usual. The path through the third SP starts in the fourth quadrant and works its way toward the positive real axis; it ends up in the ES location, but the function values get so small that computation ceases at the specified tolerance before that could happen. The path through the second SP starts near the positive real axis and ends up in the second quadrant; it starts at the ES location, but the function values are again too small. Each of these last two contours, by themselves, do not yield a useful result. However, the combination of the two paths encircles both singularities in the positive sense, leading to integral value $\mathrm{Res}(0) + \mathrm{Res}(1) = 1 - \mathrm{Q} = \mathrm{q} = 1.7196 - i\,5.2731$. Thus, here is an example where a combination of two SD paths *must* be used to attain a useful result. Taken separately, none of the three SD contours through the three SPs yield useful integral results relative to $Q_M(a,b)$ or $q_M(a,b)$. In fact, it is conceivable that in some cases, it may be necessary to use a combination of all three SPs and their respective descent paths to achieve a meaningful integral result.

The eighth case is

$$M = 1000, \; a = 1.1 + i\,3, \; b = 2.1 - i\,2,$$
$$z_s = -10.602 - i\,237.77, \; 0.00100 - i\,7.5455\mathrm{e}-6, \; 0.99611 + i\,0.0033. \qquad (196)$$

The SD paths through the first SP encircle both singularities in a negative sense. However, $\psi(z_s) = -4473.2 + i\,1.0377$ has such a large negative real part that the factor $\exp[\psi(z_s)]$ in equation (142) underflows. In this case, it is necessary to resort to the procedure mentioned in equation (144); the end result is that $\log(I) = -4472.2 + i\,2.5598$, which retains significance for further computations. Use of the second SP leads to encirclement of just the pole at the origin and a known value of 1 for the integral. The path associated with the third SP leads into the ES location and is not useful. This example illustrates that large values of $M$ can be handled by this combined SP and SD procedure, although it may be necessary to output a logarithm of the integral instead of the integral value itself.

The basic difficulty with this approach is the presence of the ES. The ES acts like a directive "black hole," attracting any nearby SD path directly into itself. The SD paths through the three SPs are adversely affected by this effect, making interpretation of these results very difficult and virtually impossible to automate. Visual inspection of all the SD paths is required in order to make the correct decision about what path(s) to use. Integrand (179) has no zeros anywhere in the finite $z$-plane; therefore, the SD paths cannot terminate in zeros, fortunately.

## AN ALTERNATIVE STEEPEST-DESCENT APPROACH FOR $Q_M(a,b)$

The presence of both a pole and an ES in the integrand $\Psi(z)$ in equation (179) causes difficulties in interpretation of the SPs and their associated paths of SD. It is desirable to have an alternative integrable form that does not have the ES. Such a form can be obtained as follows. In the top line of equation (108), let $y=-x$ in the latter half of that equation. Then, for $a \neq 0,\ b \neq 0,\ |a|<|b|$, the resultant two halves of the equation can be combined into the form

$$\begin{aligned} Q_M(a,b) &= \exp\left(-\frac{a^2+b^2}{2}\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \exp[a\,b\cos(x)] \frac{\exp[i\,x\,(M-1)]}{1-\frac{a}{b}\exp(-i\,x)} \\ &= \exp\left(-\frac{1}{2}(a-b)^2\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \frac{\exp[-a\,b\,[1-\cos(x)]+i\,x\,(M-1)]}{1-\frac{a}{b}\exp(-i\,x)}. \end{aligned} \tag{197}$$

Define the analytic continuation of the numerator of the integrand as

$$N(z)=\exp[-a\,b\,[1-\cos(z)]+i\,z\,(M-1)], \tag{198}$$

which is an entire function with no zeros in the finite $z$-plane. Also, define the denominator

$$D(z)=1-\frac{a}{b}\exp(-i\,z). \tag{199}$$

Observe that $N(0)=1,\ D(0)=1-a/b$, and

$$N(z+2\pi n)=N(z), \quad D(z+2\pi n)=D(z) \quad \text{for } n=0,\pm 1,\pm 2,\ldots; \tag{200}$$

that is, both functions have period $2\pi$ in the direction of the $x$-axis. Then, the integrand of equation (197) is

$$\Psi(z)=\frac{N(z)}{D(z)}. \tag{201}$$

There follows

$$\psi(z)=\mathrm{Log}\,\Psi(z)=-a\,b\,[1-\cos(z)]+i\,z\,(M-1)-\log\left(1-\frac{a}{b}\exp(-i\,z)\right)+i\,2\pi\,n, \tag{202}$$

where the two functions, Log( ) and log( ), are the general logarithm and principal value logarithm, respectively, while $n$ is an arbitrary integer. Also,

$$\begin{aligned}\psi'(z) &= -a\,b\sin(z) + i\,(M-1) - i\,\frac{a/b\exp(-i\,z)}{1-a/b\exp(-i\,z)} \\ &= -a\,b\sin(z) + i\,(M-1) - i\,\frac{1-D(z)}{D(z)} \\ &= -a\,b\sin(z) + i\,M - \frac{i}{D(z)},\end{aligned} \tag{203}$$

$$\psi''(z) = -a\,b\cos(z) + \frac{D(z)-1}{D^2(z)}. \tag{204}$$

The SP locations $\{z_s\}$ of the integrand $\Psi(z)$ in equation (201) are at the solutions of the equation $\psi'(z_s) = 0$. Let $e = \exp(-i\,z_s)$. Then, from equation (203),

$$-a\,b\frac{1/e-e}{i2} + i\,M - \frac{i}{1-(a/b)\,e} = 0, \tag{205}$$

which can be simplified to

$$a^2\,b\,e^3 - a\,(2M+b^2)\,e^2 + b\,(2M-2-a^2)\,e + a\,b^2 = 0. \tag{206}$$

Although this cubic in $e$ can be solved analytically, the solution is very complicated, even for small values of integer $M$. Instead, solve this cubic numerically for the three values of $e$. Then,

$$z_s = i\,\mathrm{Log}(e) = i\,\log(e) + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{207}$$

gives the locations of the three SPs in *each* vertical strip of width $2\pi$ in the $z$-plane.

The poles of integrand $\Psi(z)$ are located where $D(z_p) = 0$; from equation (199),

$$\begin{aligned}&\frac{a}{b}\exp(-i\,z_p) = 1, \quad \exp(i\,z_p) = \frac{a}{b}, \quad i\,z_p = \mathrm{Log}\left(\frac{a}{b}\right) = \log\left(\frac{a}{b}\right) + i\,2\pi n, \\ &z_p = 2\pi n + \mathrm{angle}\left(\frac{a}{b}\right) - i\,\log\left|\frac{a}{b}\right|.\end{aligned} \tag{208}$$

This pole location is independent of $M$. There is only one pole in each $2\pi$ vertical strip. And since $|a| < |b|$, this pole location lies *above* the real axis of the $z$-plane. The residue of $\Psi(z)$ at this pole location $z_p$ is

$$\frac{N(z_p)}{D'(z_p)} = \frac{\exp\left[-a\,b + a\,b\,\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)\right]\left(\frac{a}{b}\right)^{M-1}}{i\,\frac{a}{b}\,\frac{b}{a}} = \frac{1}{i}\left(\frac{a}{b}\right)^{M-1}\exp\left(\frac{1}{2}(a-b)^2\right). \tag{209}$$

Therefore, from equation (197),

$$\exp\left(-\frac{1}{2}(a-b)^2\right)\left(\frac{b}{a}\right)^{M-1}\frac{1}{2\pi}\int_C dz\,\frac{N(z)}{D(z)} = 1 \tag{210}$$

if contour $C$ encloses just the pole at $z_p$, in a positive sense.

## LOCATIONS OF THE HILLS AND VALLEYS OF INTEGRAND $\Psi(z)$

As $y \to \pm\infty$, the dominant quantity in $N(z)$ in equation (198) is

$$g(z) \equiv \exp[a\,b\cos(z)], \tag{211}$$

as will be seen below. This function has period $2\pi$. Let

$$a\,b = \mu\exp(i\beta), \tag{212}$$

where $\mu$ is positive real and $\beta$ is real. Then,

$$\begin{aligned} g(z) &= \exp\left[\frac{\mu}{2}\exp(i\beta)\,\{\exp(i\,z)+\exp(-i\,z)\}\right] \\ &= \exp\left[\frac{\mu}{2}\exp(i\beta + i\,x - y)\right]\exp\left[\frac{\mu}{2}\exp(i\beta - i\,x + y)\right]. \end{aligned} \tag{213}$$

As $y \to +\infty$, then

$$g(z) \sim \exp\left[\frac{\mu}{2}\exp[i(\beta - x)]\exp(y)\right], \tag{214}$$

which dominates $\exp[i\,z(M-1)] = \exp[(i\,x - y)(M-1)]$. Therefore, the centers of the hills of $g(z)$ at $+i\infty$ (the location of fastest growth of the magnitude) are at $x$ coordinates

$$x_{H+} = \beta = \text{Angle}(a\,b) = \text{angle}(a\,b) + 2\pi\,n, \tag{215}$$

where the two functions, Angle( ) and angle( ), are the general angle and principal-value angle, respectively. Also, the centers of the valleys of $g(z)$ at $+i\infty$ (the location of fastest decay of the magnitude) are at

$$x_{V+} = \beta + \pi = \text{angle}(a\,b) + \pi + 2\pi n. \tag{216}$$

On the other hand, as $y \to -\infty$, then

$$g(z) \sim \exp\left[\frac{\mu}{2} \exp[i(\beta + x)] \exp(-y)\right]. \tag{217}$$

Now, the centers of the hills at $-i\infty$ are at

$$x_{H-} = -\beta = -\text{Angle}(a\,b) = -\text{angle}(a\,b) + 2\pi n, \tag{218}$$

while the centers of the valleys at $-i\infty$ are at

$$x_{V-} = \pi - \beta = -\text{angle}(a\,b) + \pi + 2\pi n. \tag{219}$$

## MOVEMENT OF CONTOUR FOR $Q_M(a,b)$

The original contour of integration for $Q_M(a,b)$ in equation (197) is a finite contour along the real axis of the $z$-plane, from $-\pi$ to $\pi$. However, since the integrand of equation (197) has period $2\pi$, the range of integration can be taken as $x_c$ to $x_c + 2\pi$, where

$$x_c = \text{angle}(a\,b) - \pi; \tag{220}$$

see equation (216). Then, two vertical lines can be drawn from $x_c$ and $x_c + 2\pi$ toward $+i\infty$, right into the centers of two adjacent valleys. The combination of these two vertical lines and the portion of the real axis between them is a U-shaped contour, to be denoted by $C_U$. Integration down the left vertical line exactly cancels integration up the right vertical line, due to the periodicity of the integrand. Thus, there follows, from equations (197) through (201),

$$Q_M(a,b) = \exp\left(-\frac{1}{2}(a-b)^2\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_{C_U} dz\, \frac{N(z)}{D(z)}. \tag{221}$$

Now, infinite contour $C_U$ can be moved upward so as to pass through a convenient SP(s); call this new contour $C_s$. If the pole of the integrand at $z_p$ (which is in the upper-half $z$-plane for the present case of $|a| < |b|$) is crossed during this contour movement, then use of equation (210) reveals that

$$Q_M(a,b) = \exp\left(-\frac{1}{2}(a-b)^2\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_{C_s} dz \frac{N(z)}{D(z)} + 1. \tag{222}$$

This result is useful for evaluating complementary function $q_M(a,b) = 1 - Q_M(a,b)$. However, if the pole at $z_p$ is *not* crossed when moving contour $C_U$ into $C_s$, then the additive 1 on the right-hand side of equation (222) must be removed, and a direct integral relation for $Q_M(a,b)$ results.

In some cases, contour $C_s$ may not look like a U-shaped curve. It may be necessary to use two SPs and piece together two steepest-descent contours that enter a valley at $-i\infty$ with an $x$-coordinate midway between the locations of the original two adjacent valleys of $C_U$ at $+i\infty$. Of course, contour $C_s$ would still start in the left valley at $x_c + i\infty$ and end up in the right valley at $x_c + 2\pi + i\infty$. Whether equation (222), with or without the additive 1, should be used depends on the exact positions of the three SPs and the pole of integrand $\Psi(z)$.

The development above presumed that $|a| < |b|$, meaning that pole location $z_p$ lies in the upper-half $z$-plane. Keeping $a$ and $b$ fixed for the moment, contour $C_U$ in equation (221) can be moved downward in the $z$-plane, according to Cauchy's theorem, *without* changing the value of the integral (221) for $Q_M(a,b)$; call this new contour $C_d$. Then, since $Q_M(a,b)$ is analytic in both $a$ and $b$, the values of these two parameters can now be varied as desired, provided that the pole location does not cross $C_d$; this can always be accomplished by sufficient prior movement of $C_d$. Thus, if contour $C_d$ remains *below* the pole location, the expression

$$Q_M(a,b) = \exp\left(-\frac{1}{2}(a-b)^2\right)\left(\frac{b}{a}\right)^{M-1} \frac{1}{2\pi} \int_{C_d} dz \frac{N(z)}{D(z)} \tag{223}$$

holds true for *all* $a$ and $b$, not just $|a| < |b|$. Contour $C_d$ can *now* be moved so as to pass through an SP(s), resulting in a modified contour $C_s$. Again, relation (222) remains valid if the pole location is crossed while $C_d$ is moved into $C_s$, whereas the additive 1 on the right-hand side is dropped if the pole location is not crossed during movement. Thus, relation (222) can be used for *all* values of $a$ and $b$, provided that this "crossing rule" is adhered to. In no case does contour $C_s$ in equation (222) pass through the pole location $z_p$, because SD paths always automatically avoid poles.

## SELECTION OF AN APPROPRIATE SADDLEPOINT

Equations (206) and (207) give the locations of the three SPs of integrand $\Psi(z)$ of equation (201), while equation (208) gives the pole locations. When the SD paths through the three SPs are plotted, a wide variety of contours is obtained, making it very difficult to decide what combination to select, without making a detailed observation of the particular paths. This makes automation of the $Q_M(a,b)$ evaluation very difficult.

Typically, one SP is associated with the pole and always lies in close proximity to the pole location. The SD path through this particular SP surrounds the pole and frequently comes and goes into the *same* valley at infinity. Thus, the integral along this SD path always gives the values $\pm 1$ and is, therefore, useless. The SD paths through the two remaining SPs sometimes have to be combined, to yield a complete path leading between two adjacent valleys. Yet, on other occasions, one SP will connect two valleys at $z = \pm i\infty$, while the remaining SP will accomplish the *desired* task of connecting two adjacent valleys at $z = +i\infty$. These complications require that an alternative tack be taken.

From this point on, the complex parameters $a$ and $b$ of $Q_M(a,b)$ will be restricted so that their product is *real*. Then, equation (216) reveals that there are valleys at $+i\infty$ located at $x$-coordinates $\pm\pi$. Instead of trying to connect these two valleys at $z = +i\infty$ by means of an SD path through an SP(s), a simpler approach will be adopted. Namely, a horizontal path of integration (not the real axis) through an "appropriate" SP over the real interval $(-\pi,\pi)$ will be employed; the vertical legs of the paths reaching to $z = +i\infty$ can be dropped, because the integrals along these paths cancel each other. The appropriate SP is that one with the minimum magnitude of the *slope* of the SD path as it passes through the SP. This automatically eliminates those SPs that connect two different valleys at $z = \pm i\infty$, because the slope for those SPs is nearly vertical. (Other alternatives were tried, but none proved to be unambiguous for all the cases tried.) Of course, this horizontal path of integration is no longer the path of SD through that SP, but it is still a path of descent. The peak magnitude of the integrand on that horizontal path occurs right at the SP location, which is a very desirable condition. This horizontal path through the SP does not eliminate oscillations in the real and imaginary parts of the integrand, but it does tend to minimize oscillations. This switch in paths from SD to a horizontal path also eliminates the considerable complex numerical effort associated with determining the actual SD path through a particular SP.

Since this horizontal path of integration might intersect or come close to the pole location, indent the path with a portion of a small circle centered at the pole location, where and if necessary, thereby avoiding a close approach of the integration path to the pole. Then, use a numerical integration procedure on the horizontal path (and another numerical integration procedure on the indented circular portion if present) such as MATLAB's quad$\ell$ routine. This latter routine is an adaptive Gauss-Lobatto integration procedure with high accuracy and efficiency for smooth integrands. By breaking the integration path into two pieces (when necessary), the integrands in the two quad$\ell$ applications can be kept as smooth functions.

This procedure uses all the information about the locations of the three SPs and the pole, but avoids having to evaluate the detailed SD paths from any of the SPs. There is no analytical solution for the SD paths through any of the SPs. Examples of the integrand on the horizontal path reveal that moderate oscillations are encountered as the magnitude of the integrand decreases from its peak value at the SP; however, severe oscillations that would cause cancellations and loss of significance are not encountered. Also, underflow and overflow can be controlled by resorting to the factorization employed in equations (142) through (144).

## DETERMINING IF AN SD PATH PASSES UNDER OR OVER A POLE LOCATION

It has been seen above that a key issue in evaluating the contour integral is whether a particular SD path passes under or over a pole location in the complex $z$-plane. This determination can be accomplished without drawing all the various contours and can be automated as follows. Let the known locations of all the SPs be denoted by $\{z_s\}$, $s = 1:S$. Also, let $C_s$ denote the $s$-th SD contour, which passes through $z_s$. Then, from equation (137),

$$\psi_i(z) = \psi_i(z_s) \quad \text{for } z \text{ on } C_s, \quad s = 1:S. \tag{224}$$

The pole location, $z_p = x_p + i\,y_p$, must be known; for example, see equation (208). In order to determine if the SD contour $C_s$ passes under or over this location, set

$$\psi_i(x_p + i\,y_{ps}) = \psi_i(z_s) \tag{225}$$

and solve for $y_{ps}$. This solution must be conducted for each $s$ value of interest. If $y_{ps} > 0$, then SD contour $C_s$ passes above the pole location. If $y_{ps} < 0$, then $C_s$ passes below the pole location. Solution $y_{ps}$ can never equal zero because SD paths automatically avoid any poles.

If there are multiple real solutions for $y_{ps}$ in equation (225), it will be necessary to trace the detailed SD paths out of the SP of interest to determine *exactly* where the SD paths go. The large variation of path possibilities makes this computation unavoidable.

In terms of the auxiliary function $\phi(z)$ defined in equation (138), it is necessary to introduce a more explicit notation, namely,

$$\phi_s(z) \equiv \psi(z) - \psi(z_s) \quad \text{for } s = 1:S. \tag{226}$$

Then, the imaginary parts become

$$\phi_{si}(z) = \psi_i(z) - \psi_i(z_s) \quad \text{for } s = 1:S, \tag{227}$$

and, in particular,

$$\phi_{si}(x_p + i\,y_{ps}) = \psi_i(x_p + i\,y_{ps}) - \psi_i(z_s) \quad \text{for } s = 1:S. \tag{228}$$

Therefore, solution of equation (225) is equivalent to solving

$$\phi_{si}(x_p + i\,y_{ps}) = 0 \tag{229}$$

for $y_{ps}$ for those values of $s$ of interest.

## SUMMARY

A procedure for evaluation of the generalized $Q_M(a,b)$ function, with complex arguments $a,b,$ has been derived, which keeps underflow and overflow in check. A MATLAB routine is listed in the appendix. It is limited to the case where the product $a\ b$ of its arguments is real. However, the procedures detailed in this report could be used to extend the routine to general complex arguments.

The complementary function $q_M(a,b) = 1 - Q_M(a,b)$ often occurs in calculations for a practical problem. Accordingly, it is necessary to *also* calculate and supply this quantity. In fact, in general, the *four* quantities

$$Q_M,\ q_M,\ \log(Q_M),\ \log(q_M) \tag{230}$$

should all be output from the general routine. This guarantees that *at least one* of the four complex quantities retains significance for *all* argument values. This situation is similar to the case in MATLAB, where routines for both the error function erf( ) and the complementary error function erfc( ) are furnished. All of these features are included in the final routine for the $Q_M(a,b)$ function. In particular, the call is

$$[Q, q, Q\log, q\log, j, n] = Qq(M, a, b)\,, \tag{231}$$

where $j$ = 1 or 2 indicates whether $Q$log or $q$log is more accurate, while $n$ is the number of final evaluations utilized for MATLAB routine quad$\ell$.

A general review of steepest descent procedures, including an accurate prescription for numerically evaluating the paths of steepest descent out of a saddlepoint, has been presented. By factoring out the value of the integrand at the saddlepoint, underflow and overflow can be controlled by computing the logarithm of the specified quantity. If desired, the exponential of this logarithmic quantity can then be taken as the last step of computation; of course, this exponentiation could itself underflow or overflow.

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## APPENDIX
## PROGRAM LISTING FOR $Q_M(a, b)$

```
function [Q q Qlog qlog j n] = Qq(M,a,b) % Only for product a b real.
% QM(a,b) = int(b,inf) dx x (x/a)^(M-1) exp(-x^2/2-a^2/2) I[M-1](ax).
% qM(a,b) = 1 - QM(a,b) is the complementary QM function.
% j=1: Qlog is most accurate;   j=2: qlog is most accurate.
% n is the number of function evaluations in the calls to quadl.
global m p r cs R zp yy

tolrel=1e-6;         % Relative Tolerance on quadl integrals.
R=.2;                % Threshold for SP & pole separation.

j=1;
n=0;                 % Number of function evaluations in quadl.

if(b==0) Q=1; q=0; Qlog=0; qlog=-Inf; return, end

if(a==0)
    c=b*b/2; t=1; s=1;
    for k=1:M-1
        t=t*c/k; s=s+t;
    end
    Q=exp(-c)*s; q=1-Q; Qlog=imag_limit(-c+log(s)); qlog=log(q);
    return
end

m=M;
p=a*b;
if(abs(imag(p))>abs(real(p))*1e-14) error('a*b is not real.'), end
p=real(p);                       % Eliminate imag round-off error.
r=a/b;
zp=-i*log(r);                    % zp = pole location.

e=roots([p  2*m-2-p*r  -p-2*m*r  p*r]);
zs=-i*log(e);                    % 3 SP (saddlepoint) locations.
[d,k]=min(abs(e-r));             % [d,k]=min(abs(zs-zp));
zs(k)=[];                        % Eliminate SP closest to pole.

ang=.5*(pi-angle(psi2(zs)));     % Retain the SP
a1=min(ang(1),pi-ang(1));        % with the lower
a2=min(ang(2),pi-ang(2));        % magnitude-slope
k=1; if(a2<a1) k=2; end          % of the POSD
zs=zs(k);                        % through the SP.

cs=p*cos(zs)+i*m*zs-log(exp(i*zs)-r);
faclog=-.5*(a*a+b*b)-(m-1)*log(r)-log(2*pi);
xp=real(zp); yp=imag(zp);
xs=real(zs); ys=imag(zs); yy=ys-yp;
```

```
if(abs(yy)>=R)
    [int,n]=quadl(@f,zs-pi,zs+pi,tolrel);
else
    sq=realsqrt(R*R-yy*yy);
    x1=xp-sq; x2=xp+sq;
    z1=x2-2*pi+i*ys; z2=x1+i*ys;
    fm=max(abs(f([z1 z2])));
    if(abs(xs-xp)>=sq) fm=max(fm,1); end
    tolabs=fm*tolrel;
    [int1,n1]=quadl(@f,z1,z2,tolabs);
    tp=acos(abs(yy)/R);
    t=linspace(-tp,tp,5);    % 5 samples in angle.
    big=max(abs(g(t)));
    gtolabs=tp*big*tolrel;  % Absolute tolerance on quadl g integral.
    [int2,n2]=quadl(@g,-tp,tp,gtolabs);
    int=int1+int2; n=n1+n2;
end

if(yy<0)
    Qlog=imag_limit(faclog+cs+log(int));
    Q=exp(Qlog); q=1-Q; qlog=log(q); j=1;
else
    qlog=imag_limit(faclog+cs+log(int)+i*pi);
    q=exp(qlog); Q=1-q; Qlog=log(Q); j=2;
end
if(imag(a)==0)
    Q=real(Q); q=real(q); Qlog=real(Qlog); qlog=real(qlog);
end

% keyboard


function w = f(z)        % f(zs) = 1
global m p r cs
w=exp(p*cos(z)+i*m*z-log(exp(i*z)-r)-cs);

function w = g(t)
global m p r cs R zp yy
c=cos(t); s=sin(t);
u=R*(s+i*c);
if(yy<0) u=conj(u); s=-s; end
z=zp+u;
w=R*(c-i*s).*exp(p*cos(z)+i*m*z-log(exp(i*z)-r)-cs);

function w = psi2(z)
global p r
e=exp(i*z);
w=-p*cos(z)-r*e./(e-r).^2;
```

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